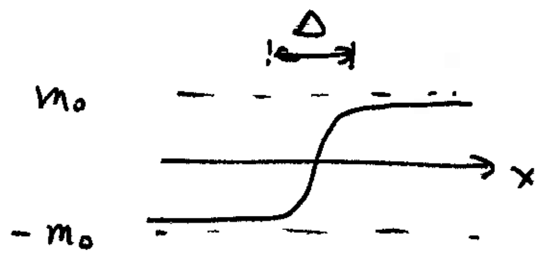
Physics 212 – Problem Set # 7
Solutions

1.) $$G[m] = \int d^3x \left\{ \frac{K}{2}(\vec{\nabla} m)^2 + \frac{g}{4!}(m^2 - m_0^2)^2 \right\}$$

a.) Consider a configuration $m(x)$ that crosses over from $m = -m_0$ to $m = +m_0$ in a distance Δ



$$\frac{G}{\text{Area}} \sim \Delta \cdot \frac{K}{2} \cdot \left(\frac{m_0}{\Delta}\right)^2 + \Delta \cdot \frac{g}{4!}(m_0^2)^2$$

up to coefficients of order 1

$$\sim \frac{K}{2}\frac{m_0^2}{\Delta} + g\, m_0^4\, \Delta$$

minimize w. resp to Δ: $\frac{\partial}{\partial \Delta}(\text{above}) = -\frac{K}{2}\frac{m_0^2}{\Delta^2} + g m_0^4$

or $$\Delta \sim \left(\frac{K}{g}\right)^{\frac{1}{2}} \frac{1}{m_0}$$

then $$\frac{G}{\text{Area}} \sim (gK)^{\frac{1}{2}} m_0^3$$

b.) The variational equation is

$$0 = \delta G = \int d^3x \left\{ K\, \vec{\nabla} m \cdot \vec{\nabla} \delta m + \frac{g}{3!} \delta m \cdot m (m^2 - m_0^2) \right\}$$

$$= \int d^3x \;\; \delta m \cdot \left\{ -K \nabla^2 m + \frac{g}{3!} m (m^2 - m_0^2) \right\}$$

For a one-dimensional configuration:

$$-K \frac{d^2}{dx^2} m + \frac{g}{3!} m (m^2 - m_0^2) = 0$$

c.) Let $$E = \frac{K}{2} \left(\frac{dm}{dx} \right)^2 - \frac{g}{4!} (m^2 - m_0^2)^2$$

then $$\frac{dE}{dx} = K \frac{dm}{dx} \frac{d^2 m}{dx^2} - \frac{g}{3!} \frac{dm}{dx} m (m^2 - m_0^2)$$

$$= -\frac{dm}{dx} \left(-K \frac{d^2 m}{dx^2} + \frac{g}{3!} m (m^2 - m_0^2) \right)$$

so $\frac{dE}{dx} = 0$ implies the variational equation

$$-K \frac{d^2 m}{dx^2} + \frac{g}{3!} m (m^2 - m_0^2) = 0$$

There is a mechanical analogy here: Write

$$m \to z \qquad x \to t \qquad -\frac{g}{4!}(m^2-m_0^2)^2 = V(z)$$

$$K \to M$$

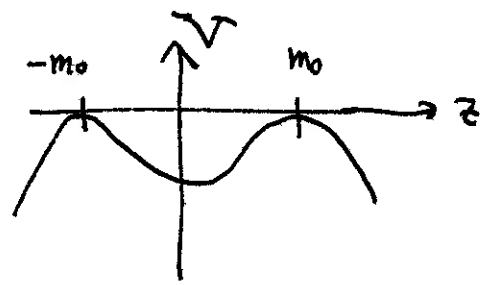
then the variational equation reads:

$$M\frac{d^2z}{dt^2} = -V'(z)$$

and
$$E = \tfrac{1}{2}M\left(\frac{dz}{dt}\right)^2 + V(z)$$

This is a mechanics problem with the potential

$$V(z) = -\frac{g}{4!}(z^2-m_0^2)^2$$



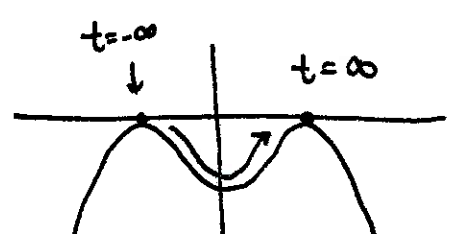
We would like to find a solution for which, as $t \to \infty$

$$z \to m_0 \qquad \text{and} \qquad \frac{dz}{dt} \to 0$$

so $E \to 0$

Since $\frac{dE}{dt} = 0$, we must have $E = 0$ for all t:

This is the solution:



which begins at the top of one hill and rolls just to the top of the other hill. Going back to the original notation:

$$\frac{K}{2}\left(\frac{dm}{dx}\right)^2 = \frac{g}{4!}(m^2-m_0^2)^2$$

$$\frac{dm}{dx} = \left(\frac{g}{12K}\right)^{1/2}(m_0^2-m^2)$$

since we want $\frac{dm}{dx}>0$ $|m|<m_0$

$$\int\frac{dm}{m_0^2-m^2} = \left(\frac{g}{12K}\right)^{1/2}\int dx$$

$$\frac{1}{m_0}\tanh^{-1}\left(\frac{m}{m_0}\right) = \left(\frac{g}{12K}\right)^{1/2}x$$

$$m(x) = m_0\tanh\left(\left(\frac{g}{12K}\right)^{1/2}m_0 x\right)$$

Indeed, the wall thickness is $\Delta = \left(\frac{12K}{g}\right)^{1/2}\frac{1}{m_0}$

Translation to the notation used in class:

$$K\to\rho \qquad m_0\to\left[\frac{a(T_c-T)}{b}\right]^{1/2} \qquad \frac{g}{6}\to b$$

$$m(x) = \left[\frac{a(T_c-T)}{b}\right]^{1/2}\tanh\left[\left(\frac{b}{2\rho}\right)^{1/2}\left[\frac{a(T_c-T)}{b}\right]^{1/2}x\right]$$

$$= m_0\tanh\left[\left[\frac{a(T_c-T)}{2\rho}\right]^{1/2}x\right] \quad \checkmark$$

2.) a) $$G[M] = \tfrac{1}{2}a(T-T_c)(M^i)^2 + \tfrac{b}{4}[(M^i)^2]^2 + \tfrac{c}{4}(M^i)^4$$

$$\frac{\partial G}{\partial M} = 0 = a(T-T_c)M^i + bM^i(M^j)^2 + c(M^i)^3$$

so for $T<T_c$

$$a(T_c-T) = b\sum_j (M^j)^2 + c(M^i)^2$$

or

$$M^i = 0$$

for each i.

We can find solutions for various direction of M^i with respect to the axes.

$M^i = (M_0, 0, 0)$

$i=1$ $$a(T_c-T) = (b+c)M_0^2 \Rightarrow M_0 = \left[\frac{a(T_c-T)}{b+c}\right]^{1/2}$$

$M^i = \frac{1}{\sqrt{2}}(M_0\ M_0, 0)$

for $i=1$ $$a(T_c-T) = bM_0^2 + \frac{c}{2}M_0^2 \Rightarrow M_0 = \left[\frac{a(T_c-T)}{b+c/2}\right]^{1/2}$$

$M^i = \frac{1}{\sqrt{3}}(M_0, M_0, M_0)$

for $i=1$ $$a(T_c-T) = bM_0^2 + \frac{c}{3}M_0^2 \Rightarrow M_0 = \left[\frac{a(T_c-T)}{b+c/3}\right]^{1/2}$$

Compute the values of G for these solutions:

$$M = (M_0, 0, 0)$$

$$G = \frac{1}{2} a (T - T_c) \left[\frac{a(T_c - T)}{b+c} \right] + \left(\frac{b+c}{4} \right) \left[\frac{a(T_c - T)}{b+c} \right]^2$$

$$= -\frac{1}{4} \frac{a^2}{b+c} (T_c - T)^2$$

$$M = \frac{1}{\sqrt{2}} (M_0, M_0, 0)$$

$$G = \frac{1}{2} a (T - T_c) \frac{a(T_c - T)}{(b + \frac{c}{2})} + \frac{b}{4} \left[\frac{a(T_c - T)}{b + c/2} \right]^2$$

$$+ \frac{c}{4} \, 2 \cdot \frac{M_0^4}{4}$$

$$= -\frac{1}{4} \frac{a^2}{(b + c/2)} (T_c - T)^2$$

$$M = \frac{1}{\sqrt{3}} (M_0, M_0, M_0)$$

$$G = \frac{1}{2} a (T - T_c) \left[\frac{a(T_c - T)}{b + c/3} \right] + \frac{b}{4} \left[\frac{a(T_c - T)}{b + c/3} \right]^2$$

$$+ \frac{c}{4} \cdot 3 \cdot \frac{1}{9} \left[\frac{a(T_c - T)}{b + c/3} \right]^2$$

$$= -\frac{1}{4} \frac{a^2}{b + c/3} (T_c - T)^2$$

so for $c < 0$ $\quad (b > c)$

the solution $M = (M_0, 0, 0)$ is preferred

actually
$$M = (\pm M_0, 0, 0)$$
$$(0, \pm M_0, 0)$$
$$(0, 0, \pm M_0)$$
give 6 degenerate solutions

for $c > 0$

the solutions

$$M = \frac{1}{\sqrt{3}}(\pm M_0, \pm M_0, \pm M_0)$$

give 8 degenerate solutions.

b.) For the ground state $(M_0, 0, 0)$ $\quad \underline{c < 0}$

write

$$M^i = (M_0 + m, \pi^2, \pi^3)$$

$$(M^i)^2 = (M_0 + m)^2 + (\pi^k)^2 \qquad k = 2, 3$$

$$= M_0^2 + 2M_0 m + m^2 + (\pi^k)^2$$

$$((M^i)^2)^2 = M_0^2 + 4M_0^3 m + 6M_0^2 m^2 + \cdots$$
$$+ 2M_0^2 (\pi^k)^2 + \cdots$$

$$(M^i)^4 = M_0^2 + 4M_0^3 m + 6M_0^2 m^2 + \mathcal{O}(m^3)$$
$$+ \mathcal{O}(\pi^4)$$

Then the Gibbs free energy for spin densities is

$$G = \int d^3x \Big\{ \tfrac{1}{2}\rho(\vec\nabla m)^2 + \tfrac{1}{2}\rho(\vec\nabla \pi^k)^2 \quad + \quad (\text{indep of } m, \pi)$$

$$+ \tfrac{1}{2}a\,(T-T_c)\left[2M_0 m + m^2 + (\pi^k)^2\right]$$

$$+ \quad \frac{b}{4}\left[4M_0^3 m + 6M_0^2 m^2 + 2M_0^2(\pi^k)^2 + \cdots\right]$$

$$+ \quad \frac{c}{4}\left[4M_0^3 m + 6M_0^2 m^2 + \cdots\right] \Big\}$$

$$= \int d^3x \Big\{ \tfrac{1}{2}\rho(\vec\nabla m)^2 + \tfrac{1}{2}\rho(\vec\nabla\pi^k)^2$$

$$= 0 \longrightarrow \quad + \left[-a(T_c-T)M_0 + (b+c)M_0^3\right] m$$

$$+ \left[-\tfrac{1}{2}a(T_c-T) + \tfrac{3}{2}M_0^2(b+c)\right] m^2$$

$$\underline{\underline{\text{not}}} = 0 \longrightarrow \quad + \left[-\tfrac{1}{2}a(T_c-T) + \tfrac{1}{2}b\,M_0^2\right](\pi^k)^2$$

$$+ \cdots \Big\}$$

$$= \int d^3x \Big\{ \tfrac{1}{2}\rho(\vec\nabla m)^2 + \tfrac{1}{2}\cdot\left[2\;a(T_c-T)\right]m^2$$

$$+ \tfrac{1}{2}\rho(\vec\nabla\pi^k)^2 + \tfrac{1}{2}\left(\frac{b}{b+c} - 1\right)a(T_c-T)\,(\pi^k)^2 \Big\}$$

$$= \int d^3x \Big\{ \tfrac{1}{2}\rho(\vec\nabla m)^2 + \tfrac{1}{2}\,2a(T_c-T)\,m^2$$

$$+ \tfrac{1}{2}\rho(\vec\nabla\pi^k)^2 + \tfrac{1}{2}\left(\frac{-c}{b+c}\right)a(T_c-T)(\pi^k)^2 \Big\}$$

the variational equations for m and π are

$$-\nabla^2 m + \frac{2a(T_c - T)}{\rho}\, m = 0$$

$$-\nabla^2 \pi^k + \left(\frac{-c}{b+c}\right) \frac{a(T_c - T)}{\rho}\, \pi^k = 0$$

then, as we discussed in class, the correlation functions are the Green's function of these equations:

$$\langle m(x)\, m(0)\rangle \sim \frac{1}{4\pi r} \exp\left[-r/\xi_m(T)\right]$$

$$\langle \pi^k(x)\, \pi^l(0)\rangle \sim \frac{\delta^{kl}}{4\pi r} \exp\left[-r/\xi_\pi(T)\right]$$

With $\xi_m(T) = \left[\frac{\rho}{2a(T_c - T)}\right]^{1/2}$ $\quad \xi_\pi(T) = \left[\left[\frac{b+c}{-c}\right] \frac{\rho}{a(T_c - T)}\right]^{1/2}$

$$\begin{aligned}\langle s^i(x)\, s^j(0)\rangle = {} & M_0^2\, \delta^{i1}\delta^{j1} \\ & + \frac{1}{4\pi r} \exp\left[-r/\xi_m(T)\right] \delta^{i1}\delta^{j1} \\ & + \frac{1}{4\pi r} \exp\left[-r/\xi_\pi(T)\right] \delta^{ik}\underbrace{\delta^{jk}}_{k=2,3}\end{aligned}$$

When $c \to 0$, the π^k become Goldstone bosons and $\xi_\pi^{-1}(T) \to 0$.

c.) For $c>0$, we must expand about

$$M = \tfrac{1}{\sqrt{3}}(M_0, M_0, M_0)$$

There are easier ways, but let me do this by brute force

$$M = \left(\tfrac{1}{\sqrt{3}}M_0 + m^1,\ \tfrac{1}{\sqrt{3}}M_0 + m^2,\ \tfrac{1}{\sqrt{3}}M_0 + m^3\right)$$

$$(M^i)^2 = \tfrac{1}{3}M_0^2 + \tfrac{2}{\sqrt{3}}M_0 m^1 + (m^1)^2 + \tfrac{1}{3}M_0^2 + \tfrac{2}{\sqrt{3}}M_0 m^2 + (m^2)^2$$
$$+ \tfrac{1}{3} + \tfrac{2}{\sqrt{3}}M_0 m^3 + (m^3)^2$$

$$= M_0^2 + 2M_0 \tfrac{1}{\sqrt{3}}(m^1+m^2+m^2) + (m^1)^2 + (m^2)^2 + (m^3)^2$$

$$((M^i)^2)^2 = M_0^4 + 4M_0^3 \tfrac{1}{\sqrt{3}}(m^1+m^2+m^3) + \tfrac{4}{3}M_0^2(m^1+m^2+m^3)^2$$
$$+ 2M_0^2\left[(m^1)^2+(m^2)^2+(m^3)^2\right] + \mathcal{O}(m^3)$$

$$(M^i)^4 = 3\cdot\tfrac{1}{9}M_0^4 + \tfrac{4}{\sqrt{3}}\tfrac{M_0^3}{3}(m^1+m^2+m^3) + \tfrac{6}{3}M_0^2\left((m^1)^2+(m^2)^2+(m^3)^2\right)$$
$$+ \mathcal{O}(m^3)$$

$$\tfrac{1}{2}a(T-T_c)(M^i)^2 + \tfrac{b}{4}((M^i)^2)^2 + \tfrac{c}{4}(M^i)^4 \qquad = 0$$

$$= (\text{indep of } m) + \left(\tfrac{m^1+m^2+m^3}{\sqrt{3}}\right)\left[-a(T_c-T)M_0 + bM_0^3 + \tfrac{c}{3}M_0^3\right]$$
$$+ \tfrac{1}{2}\left[(m^1)^2+(m^2)^2+(m^3)^2\right]\left[-a(T_c-T) + bM_0^2 + c\,M_0^2\right]$$
$$+ \tfrac{b}{3}(m^1+m^2+m^3)^2\cdot M_0^2 + \mathcal{O}(m^3)$$

$$= (\text{indep of } m) + \frac{1}{2}\left[(m^1)^2+(m^2)^2+(m^3)^2\right]\left(\underbrace{\frac{b+c}{b+c/3}-1}_{\frac{2}{3}\frac{c}{b+c/3}}\right) a(T_c-T)$$

$$+ \frac{1}{3}(m^1+m^2+m^3)^2 \frac{a(T_c-T)}{b+c/3} + O(m^3)$$

$$= \frac{1}{2}(m_1\ m_2\ m_3)\ \underline{X} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

where $$\underline{X} = \frac{a(T_c-T)}{b+c/3}\left(\begin{bmatrix} 2/3c & & \\ & 2/3c & \\ & & 2/3c \end{bmatrix} + \frac{2}{3}\begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix}\right)$$

$$= \frac{a(T_c-T)}{b+c/3} \cdot \frac{2}{3}\ \underline{Y}$$

where $$\underline{Y} = \begin{pmatrix} b+c & b & b \\ b & b+c & b \\ b & b & b+c \end{pmatrix}$$

We ought to diagonalize Y. One eigenvector is obviously

$$\xi_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda = 3b+c$$

The orthogonal vectors

$$\xi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \xi_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad \text{have} \quad \lambda = c$$

then

$$\mathbb{X}\,\xi_1 = 2a(T_c - T)\,\xi_1$$

$$\mathbb{X}\,\xi_2 = 2a(T_c - T)\left(\frac{c/3}{b+c/3}\right)\xi_2$$

similarly for ξ_3

write

$$\xi_m = \left[\frac{\rho}{2a(T_c - T)}\right]^{1/2} \qquad \text{as on p. 9}$$

$$\xi_\pi = \left[\frac{b+c/3}{c/3}\,\frac{\rho}{2a(T_c - T)}\right]^{1/2}$$

then

$$\frac{\mathbb{X}}{\rho} = |\xi_1)\,\frac{1}{\xi_m^2}\,\langle\xi_1| + \sum_{k=2,3} |\xi_k)\,\frac{1}{\xi_\pi^2}\,\langle\xi_k|$$

The Gibbs free energy for spin densities is

$$G[M] = \int d^3x \left\{ \frac{1}{2}\rho(\vec\nabla m^i)^2 + \frac{1}{2}(m_1 m_2 m_3)\,\mathbb{X}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \right\}$$

the variational equation is

$$-\nabla^2 m^i + \frac{\mathbb{X}^{ij}}{\rho}\,m_j = 0$$

whose solutions are

$$|m^i\rangle = \frac{1}{4\pi r}\,e^{-r/\xi_a}\cdot|\xi\rangle$$

for each eigenvector.

the spin-spin correlation function is

$$\langle S^i(x)\, S^j(0) \rangle = \frac{1}{3}(M_0\, M_0\, M_0)^i (M_0\, M_0, M_0)^j$$

$$+ \frac{1}{4\pi r} e^{-r/\xi_m} \quad \xi_1^i\, \xi_1^j$$

$$+ \frac{1}{4\pi r} e^{-r/\xi_\pi} \left[\xi_2^i\, \xi_2^j + \xi_3^i\, \xi_3^j \right]$$

again, as $c \to 0$ $\quad \xi_\pi^{-1} \to 0$ and the eigenvectors $\xi_2\, \xi_3$ become Goldstone bosons.

d.) To compute the magnetic susceptibility, we first solve for h^i to linear order in M^i by solving

$$\frac{\partial G}{\partial M^i} = h^i$$

For $c<0$

$$G = \frac{1}{2} \cdot 2a(T_c - T)\, m^2 + \frac{1}{2}\left(\frac{-c}{b+c}\right) a(T_c - T)(\pi^k)^2 + \cdots$$

so

$$\frac{\partial G}{\partial m} = 2a(T_c - T)\, m = \rho \frac{1}{\xi_m^2}\, m$$

$$\frac{\partial G}{\partial \pi^k} = \left(\frac{-c}{b+c}\right) a(T_c - T)\, \pi^k = \rho \frac{1}{\xi_\pi^2}\, \pi^k$$

so
$$h^i = \rho X^{ij} m^j \qquad m^j = (m, \pi^2 \pi^3)$$
$$m^i = \frac{1}{\rho}(X^{-1})^{ij} h^j$$
$$\left.\frac{\partial m^i}{\partial h^j}\right|_{h=0} = \frac{1}{\rho}(X^{-1})_{ij}$$

Then
$$\chi^{ij} = \frac{1}{\rho}\begin{pmatrix} \xi_m^2 & & \\ & \xi_\pi^2 & \\ & & \xi_\pi^2 \end{pmatrix}$$

Similarly, for $c>0$
$$\chi^{ij} = \frac{1}{\rho}(X^{-1})^{ij}$$
$$= \frac{\xi_m^2}{\rho}\, \xi_1^i \xi_1^j + \frac{\xi_\pi^2}{\rho}\left(\xi_2^i \xi_2^j + \xi_3^i \xi_3^j\right)$$

as $c \to 0$, in both cases, this becomes a matrix that equals
$$\frac{1}{2a(T_c - T)} \quad \text{along the direction of magnetization}$$
and $\to \infty$ as $c \to 0$ for orthogonal directions.

3.) Look for a configuration with $\vec{A} = 0$ and

$$\Phi(z) = \hat{\Phi}\, e^{ikz}$$

$$G = \int d^3x \left\{ \frac{1}{2m_*} \hbar^2 k^2 \hat{\Phi}^2 + \frac{1}{2} a (T - T_c) \hat{\Phi}^2 + \frac{b}{4} \hat{\Phi}^4 \right\}$$

minimize w. resp. to $\hat{\Phi}$:

$$\frac{\hbar^2 k^2}{m_*} \hat{\Phi} + a (T - T_c) \hat{\Phi} + b \hat{\Phi}^3 = 0$$

$$\hat{\Phi} = \left[\frac{(a(T_c - T) - \hbar^2 k^2 / m_*)}{b} \right]^{\frac{1}{2}}$$

now $\quad j = e_* |\hat{\Phi}|^2 v_s$

so

$$j = e_* \frac{(a(T_c - T) - \hbar^2 k^2 / m_*)}{b} \frac{\hbar k}{m_*}$$

maximize j with respect to k:

$$\frac{\partial j}{\partial k} = 0 = \frac{e_* \hbar}{b m_*} \left(a(T_c - T) - 3 \frac{\hbar^2 k^2}{m_*} \right)$$

so $\quad \dfrac{\hbar^2 k^2}{m_*} = \dfrac{1}{3} a (T_c - T) \qquad \hbar k = \left[\dfrac{1}{3} a m_* (T_c - T) \right]^{\frac{1}{2}}$

then

$$j = \frac{e_*}{b} \cdot \frac{2}{3} \cdot a(T_c - T) \cdot \left[\frac{1}{3}\frac{a}{m_*}(T_c - T)\right]^{1/2}$$

$$= \frac{2e_*}{b}\frac{1}{m_*^{1/2}}\left[\frac{a(T_c-T)}{3}\right]^{3/2}$$

For a wire of radius R:

$$I = \pi R^2 j = \pi R^2 \cdot \frac{2e_*}{b}\frac{1}{m_*^{1/2}}\left[\frac{a(T_c-T)}{3}\right]^{3/2}$$

The surface magnetic field is

$$B_s = \frac{2I}{cR} = \frac{4\pi R e_*}{bc\, m_*^{1/2}}\left[\frac{a(T_c-T)}{3}\right]^{3/2}$$

now penetration depth: critical field

$$\lambda = \left[\frac{4\pi e_*^2 a(T_c-T)}{b m_* c^2}\right]^{-1/2} \qquad H_c = \left(\frac{2\pi}{b}\right)^{1/2} a(T_c-T)$$

so $$B_s = \left[\frac{2\pi}{b}\right]^{1/2} a(T_c-T) \cdot \left[\frac{4\pi e_*^2}{b m_* c} a(T_c-T)\right]^{1/2}\left(\frac{2}{27}\right)^{1/2} R$$

or $$B_s = \left(\frac{2}{27}\right)^{1/2}\frac{R}{\lambda} H_c$$

(and $R \ll \lambda$ is needed to justify the assumption at j = constant across the wire)