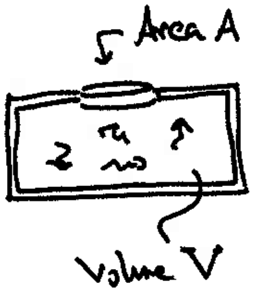
Physics 212 – Problem Set #5
Solutions

1.)



At the boundary of the hole, the photon number distribution is given by

$$N = \frac{2V}{(2\pi)^3} \int_{p_z > 0} d^3p \; \frac{1}{e^{\beta c p} - 1}$$

or

$$\frac{N}{V} = \frac{1}{4\pi^3} \int_{p_z > 0} d^3p \; \frac{1}{e^{\beta c p} - 1}$$

a.) $$\int d^3p = \int d\phi \, d\theta \sin\theta \, dp \, p^2 = 2\pi \int d(\cos\theta) \, dp \, p^2$$

$$v^z = \frac{p^z}{p} \cdot c \cdot = c \cos\theta$$

so $$\frac{N}{V} = \int_0^1 \frac{dv^z}{c} \underbrace{\int \frac{dp}{2\pi^2} \frac{p^2}{e^{\beta c p} - 1}}_{\text{this probability density is independent of } v^z}$$

b.) the total energy in the box is

$$E = 2V \int \frac{d^3p}{(2\pi)^3\hbar^3} \frac{cp}{e^{\beta cp}-1}$$

$$= \frac{V}{\pi^2} \frac{1}{\beta^4} \frac{1}{c^3\hbar^3} \int_0^\infty dx \frac{x^3}{e^x-1}$$

$$= \frac{V}{\pi^2} \frac{T^4}{c^3\hbar^3} \cdot \frac{\pi^4}{15}$$

$$E = V \cdot \frac{\pi^2 T^4}{15\, c^3\hbar^3}$$

the energy emitted through the area A in time dt

$$dE = 2 \int \frac{d^3p}{(2\pi)^3\hbar^3} \frac{cp}{e^{\beta cp}-1} \cdot A \cdot v^z\, dt \cdot \theta(v^z)$$

$$= A \cdot dt \cdot \frac{2}{8\pi^3\hbar^3} \int_0^1 2\pi\, d(\cos\theta) \int_0^\infty dp\, p^2 \cdot \frac{cp}{e^{\beta cp}-1} \cdot c \cdot \cos\theta$$

$$= A \cdot dt \cdot \frac{1}{2\pi^2\hbar^3} \left[\int_0^1 d(\cos\theta) \cos\theta\right] \int_0^\infty dp\, p^2 \frac{p}{e^{\beta cp}-1} \cdot c^2$$

$$= A \cdot dt \cdot \frac{1}{2\pi^2\hbar^3} \cdot \frac{1}{2} \cdot \frac{1}{(\beta c)^4} \cdot c^2 \cdot \frac{\pi^4}{15}$$

$$\frac{dE}{dt} = A \cdot \frac{c}{4} \cdot \left(\frac{\pi^2 T^4}{15 c^3\hbar^3}\right)$$

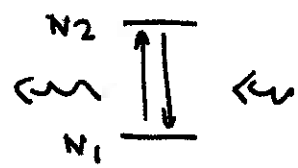
that is $\frac{dE}{dt} = \frac{1}{4} \cdot c \cdot A \cdot \left(\frac{\text{Energy in the box}}{V}\right)$

c.) Then if we write

$$\frac{dE}{dt} = A \cdot \sigma \cdot T^4$$

we have $\sigma = \dfrac{\pi^2}{60\,\hbar^3 c^2}$ this is the Stefan-Boltzmann constant, as defined in class.

2.)



$\hbar\omega = cp$

a) Absorption of photons, transition up from 1 to 2:

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} = 2\pi B\, u(\omega) \cdot N_1$$

where $u(\omega)$ is related to $n(p)$ by

$$\frac{N}{V} = \int d\omega\, u(\omega) = \int \frac{dp}{\pi^2} \frac{p^3}{e^{\beta cp}-1} = \int d\omega \frac{1}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\omega}-1}$$

Emission of photons, transition down from 2 to 1

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = A N_2$$

the total rate of chg of N_1 would be

$$\frac{dN_1}{dt} = A N_2 - 2\pi B\, u(\omega)\, N_1$$

in equilibrium $\frac{dN_1}{dt} = 0$

$$\frac{N_2}{N_1} = \frac{2\pi B}{A} \cdot u(\omega) \neq e^{-\beta\omega}$$

b.) add a third term — stimulated emission.

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = 2\pi B'\, u(\omega)\, N_2$$

then, in all

$$\frac{dN_1}{dt} = [A + 2\pi B'\, u(\omega)]\, N_2 - 2\pi B\, u(\omega)\, N_1$$

$$\frac{N_2}{N_1} = \frac{2\pi B\, u(\omega)}{A + 2\pi B'\, u(\omega)} = e^{-\beta\omega}$$

$$= \frac{2\pi B \cdot (\omega^3/\pi^2 c^3) \cdot \frac{1}{e^{\beta\omega} - 1}}{A + 2\pi B' \left(\frac{\omega^3}{\pi^2 c^3}\right) \frac{1}{e^{\beta\omega} - 1}}$$

let $A = 2\pi \cdot \frac{\omega^3}{\pi^2 c^3} \cdot a$, then we have

$$= \frac{B}{a \cdot (e^{\beta\omega} - 1) + B'} = e^{-\beta\omega}$$

then $\qquad B' = B = a \quad$ gives the solution for all ω, β

$$B = B' = A \Big/ \left(\frac{2}{\pi}\frac{\omega^3}{c^3}\right)$$

c.) The emission rate is now

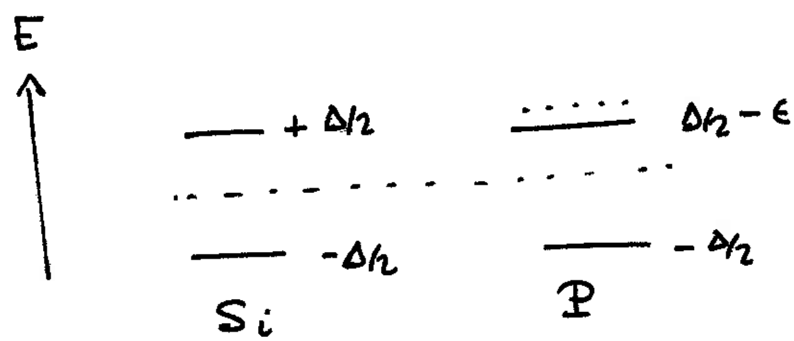
$$\frac{dN_2}{dt} = \left[A + 2\pi B'\, u(\omega)\right] N_2$$

$$\parallel$$

$$A + A\,\frac{1}{(e^{\beta\omega}-1)} = A\,(1+n(\omega))$$

$$\underset{\text{also}}{=} A \cdot \frac{e^{\beta\omega}}{e^{\beta\omega}-1}$$

3.)



a.) $$(\#\text{ of electrons}) = N \cdot \frac{1}{e^{\beta(-\Delta/2-\mu)}+1} + M\,\frac{1}{e^{\beta(\Delta/2-\epsilon-\mu)}+1}$$

$$+ (N-M)\,\frac{1}{e^{\beta(\Delta/2-\mu)}+1}$$

and $\quad \#$ of electrons $= N+M$

for each β or T, we must solve this eq. for μ

b.) $T \to \infty$ $\quad \beta\Delta \to 0$. However $\beta\mu$ must remain $O(1)$

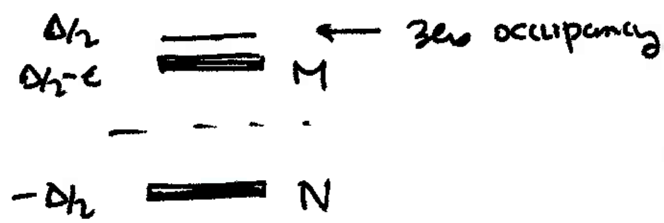
$$N+M = (N+M+N-M)\frac{1}{e^{-\beta\mu}+1}$$

$$\frac{N+M}{2N} = \frac{1}{1+e^{-\beta\mu}}$$

$$1+e^{-\beta\mu} = \frac{2N}{N+M}$$

$$e^{-\beta\mu} = \frac{N-M}{N+M} \quad \text{or} \quad \mu = +\frac{1}{\beta}\log\left(\frac{N+M}{N-M}\right)$$

c.) At $T=0$ we have:



so $\mu(T) = \left(\frac{\Delta}{2}-\epsilon\right) + \delta \qquad 0<\delta<\epsilon$

at low T, $\quad \epsilon \sim T \ll \Delta/2$,

$\mu(T)$ will still be close to $\Delta/2$.

With this value, the occupancy of the valence states at $-\Delta/2$ is

$$\frac{1}{e^{\beta(-\Delta/2-\mu)}+1} = \frac{1}{e^{-\beta\Delta}+1} = 1 \quad \text{since } e^{-\beta\Delta} \ll 1$$

N electrons occupy this state, so we are left with M electrons in the higher states.

then

$$M = \frac{M}{e^{\beta(\frac{\Delta}{2} - \epsilon - \mu)} + 1} + \frac{N-M}{e^{\beta(\Delta/2 - \mu)} + 1}$$

write $\mu = \Delta/2 + \delta\mu$

$$1 = \frac{1}{e^{-\beta\epsilon} e^{-\beta\delta\mu} + 1} + \frac{(N/M - 1)}{e^{-\beta\delta\mu} + 1}$$

write $x = e^{-\beta\delta\mu}$; this is a quadratic equation for x

$$1 = \frac{1}{x e^{-\beta\epsilon} + 1} + \frac{(N/M - 1)}{x + 1}$$

$$\frac{x e^{-\beta\epsilon}}{x e^{-\beta\epsilon} + 1} = \frac{(N/M - 1)}{x + 1}$$

$$x^2 e^{-\beta\epsilon} + x e^{-\beta\epsilon} = (N/M - 1)\left[x e^{-\beta\epsilon} + 1\right]$$

$$\left[x^2 - (N/M - 2)\, x\right] e^{-\beta\epsilon} = (N/M - 1)$$

finally $$x = \tfrac{1}{2}\left(\tfrac{N}{M} - 2\right) + \left[\tfrac{1}{4}\left(\tfrac{N}{M} - 2\right)^2 + \left(\tfrac{N}{M} - 1\right) e^{\beta\epsilon}\right]^{\frac{1}{2}}$$

$x > 0$ so only the $+$ solution is acceptable.

There are two interesting limits. First,

$N/M \gg 1$ $\quad e^{\beta\epsilon} \sim O(1)$

then $$e^{-\beta \delta\mu} = \frac{N}{M}$$

or $$\mu \cong \frac{\Delta}{2} - T \log N/M$$

Second $e^{\beta\epsilon} \gg N/M$ (which will be true as $T \to 0$)

$$e^{-\beta \delta\mu} = \left[N/M \; e^{\beta\epsilon} \right]^{1/2}$$

then $$\mu = \frac{\Delta}{2} - \frac{\epsilon}{2} - \frac{T}{2} \log \frac{N}{M}$$

d.) The condition are

$$\beta = (0.025 eV)^{-1} \qquad \epsilon = 0.044 eV \qquad \frac{N}{M} = 5\times10^{6}$$

so the first limit in (c) applies.

$$\mu = \frac{\Delta}{2} - T \log N/M$$

$$e^{\beta(\frac{\Delta}{2}-\mu)} = \frac{N}{M} = 5\times10^{6} \qquad e^{-\beta\epsilon} = 0.17$$

The occupancies are:

impurity band: $$\frac{1}{e^{\beta\epsilon}\, e^{\beta(\frac{\Delta}{2}-\mu)} + 1} = \frac{1}{8.6\times10^{5}+1} = 1.16\times10^{-6}$$

conduction band: $$\frac{1}{e^{\beta(\frac{\Delta}{2}-\mu)}+1} = 2\times10^{-7}$$

so actually most of the Phosphorus atoms are ionized, only a fraction 1.2×10^{-6} are not.

The occupancy of the valence band is

$$\frac{1}{e^{-\beta\Delta} e^{\beta(\frac{\Delta}{2}-\mu)} + 1} = 1 - e^{-\beta\Delta} e^{\beta(\frac{\Delta}{2}-\mu)} + \cdots$$

$$= 1 - \frac{N}{M} e^{-\beta\Delta} + \cdots$$

but $\beta\Delta = \frac{1.16\,eV}{.025\,eV} = 46.4$

$$e^{-\beta\Delta} = 7.1 \times 10^{-21}$$

so the fraction of unoccupied valence states is

3.5×10^{-41} or 1.8×10^{3} holes/cm^3

4.) a.) Rb^{87} has 37 p 37 e 50 n

in all, an <u>even</u> number of fermionic components

So, it is a <u>boson</u>.

b.) For free bosons in a large box, the Bose-Einstein condensation temperature is

$$k_B T_c = \frac{2\pi\hbar^2}{m} \left[\frac{N}{V} \frac{1}{\zeta(3/2)} \right]^{2/3} \qquad \zeta(3/2) = 2.612$$

$m(Rb) = 87\, m_p \sim 1.5 \times 10^{-22}\, g$

$$\frac{2\pi\hbar^2}{m} = 4.7 \times 10^{-32}\ \text{erg-cm}^2$$

for this experiment $\frac{N}{V} \sim 2.5 \times 10^{12}/cm^3$

so $k_B T_c = 4.5 \times 10^{-24}$ erg

this gives us the estimate of T_c: 3.3×10^{-8} °K $=$ 33 nK

c.) $\tilde{\psi}(p) \propto e^{-p^2/2m\hbar\omega}$

so the axis with highest frequency gives the broadest p distribution. This would correspond to the y axis in the figure.

In classical statistical mechanics, equipartition gives

$$\langle \frac{p_i^2}{2m} \rangle = \frac{1}{2} T$$

independent of the potential. Then the velocity distribution would be isotropic.

d.) For $T_{c,\,meas.} = 170$ nK

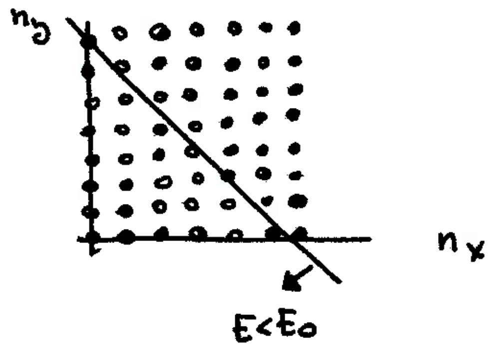
$$k_B T_c = 2.3 \times 10^{-23} \text{ erg}$$
$$\hbar\omega_0 = 7.9 \times 10^{-25} \text{ erg}$$
$$\hbar\omega_1 = 2.8 \times 10^{-25} \text{ erg}$$

so the harmonic oscillator states are relatively close compared to $k_B T_c$

e.) For a 2-d isotropic harmonic oscillator,

$$E = \hbar\omega_1 (n_x + n_y) + \hbar\omega$$

The states with $E < E_0$ are:



Ignoring the zero-point energy and assuming $E_0 \gg \hbar\omega_1$

$$(\#\text{ of states w. } E<E_0) = \int_0^\infty dn_x \int_0^\infty dn_y \; \theta(E_0 - \hbar\omega_1 (n_x+n_y))$$

$$= \tfrac{1}{2} N^2 \qquad N = \frac{E_0}{\hbar\omega_1}$$

For the 3-d anisotropic situation:

$(\#\text{ of states w. energy} < E_0)$

$$= \int dn_z \int dn_x \, dn_y \; \theta(E_0 - \hbar\omega_0 n_z - \hbar\omega_1 (n_x+n_y))$$

$$= \int_0^{E_0/\hbar\omega_0} dn_z \; [\,\#\text{ of states w. } E_{x,y} < (E_0 - \hbar\omega_0 n_z)\,]$$

$$= \int_0^{E_0/\hbar\omega_0} dn_z \; \frac{1}{2}\left[\frac{E_0 - \hbar\omega_0 n_z}{\hbar\omega_1}\right]^2$$

$$= \frac{1}{2(\hbar\omega_1)^2}\frac{1}{\hbar\omega_0} E_0^3 \left(1 - \frac{2}{2} + \frac{1}{3}\right) = \frac{E_0^3}{6(\hbar\omega_1)^2(\hbar\omega_0)}$$

Alternatively, this is the volume of a triangular region in the $n_x n_y n_z$ space, divided by the spacing of points $(\hbar\omega_1)^2(\hbar\omega_0)$

The density of states is

$$g(E_0) = \frac{d}{dE_0}(\#) = \frac{E_0^2}{2(\hbar\omega_0)(\hbar\omega_1)^2}$$

f.) The number of bosons that the well can hold at temperature T without Bose-Einstein condensation is

$$N = \int dE \; g(E) \; \frac{1}{e^{\beta E}-1}$$

$$= \int dE \; \frac{E^2}{2(\hbar\omega_0)(\hbar\omega_1)^2} \; \frac{1}{e^{\beta E}-1}$$

$$= \frac{1}{2}\frac{1}{(\hbar\omega_0)}\frac{1}{(\hbar\omega_1)^2}\frac{1}{\beta^3} \underbrace{\int_0^\infty dx \; \frac{x^2}{e^x-1}}_{\to \; = 2\zeta(3)}$$

$$N = \zeta(3)\frac{T^3}{(\hbar\omega_0)(\hbar\omega_1)^2}$$

Under the conditions described, this equals the total number of particles in the system for

$$(k_B T_c)^3 = N \frac{1}{\zeta(3)} (\hbar\omega_0)(\hbar\omega_1)^2$$

$$= (1.0 \times 10^{-23} \text{erg})^3$$

or $T_c = 73 \text{ nK}$

Presumably the observed T_c is higher because the Rb atoms have a weak attraction.

5.) a.) For a sphere of uniform density of electrons and He^4 nuclei.

① Compute the degeneracy energy of the electrons:

$$n(e^-) = \frac{N}{\frac{4}{3}\pi R^3} = \frac{1}{3\pi^2} p_F^3$$

so $$p_F = \left(\frac{9\pi}{4}\frac{N}{R^3}\right)^{1/3} \qquad \epsilon_F = \frac{\hbar^2}{2m}\left(\frac{9\pi N}{4R^3}\right)^{2/3}$$

The total energy of the system is

$$E = \frac{3}{5} N \varepsilon_F = \frac{3}{5}\frac{\hbar^2}{2m_e}\left(\frac{9\pi}{4}\right)^{2/3} \frac{N^{5/3}}{R^2}$$

② Compute the gravitational energy of the sphere. The mass density is

$$\rho = \frac{N}{2} \cdot m(He^4)$$

The energy gain from assembly a mass distribution is

$$E_G = \frac{1}{2}\int d^3x \, \rho(x) \, V_G(x)$$

where $V_G(x)$ is the Newtonian potential. For a spherical shell of matter

$$V_G(x) = -\frac{G_N M}{R} \quad \text{on the shell}$$

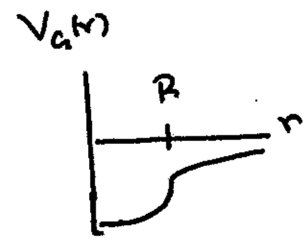
so $$E_G = -\frac{1}{2}\frac{G_N M^2}{R}$$

For a uniform sphere of matter, the potential on the surface is $\quad -\frac{G_N M}{R}$

Inside, the $\dfrac{\text{Force}}{m} = -\dfrac{G_N}{r^2} M(r) = -\dfrac{G_N}{r^2} M \dfrac{r^3}{R^3}$

$$= -\frac{G_N M r}{R^3}$$

so $\quad V_G(r) = \dfrac{G_N M}{R^3} \dfrac{r^2}{2} + \text{const.}$



then

$$V_G(r) = -\frac{G_N M}{R} + \frac{G_N M}{2R^3}(r^2 - R^2)$$

$$E_G = \frac{1}{2}\int d^3r\, \rho \left[-\frac{3}{2}\frac{G_N M}{R} + \frac{G_N M}{2R^3} r^2 \right]$$

$$= \frac{1}{2}\left[-\frac{3}{2}\frac{G_N M}{R}\left(\rho \cdot \frac{4}{3}\pi R^3\right) + \frac{G_N M}{2R^3} \cdot \left(\rho \cdot \frac{4\pi}{5} R^5\right) \right]$$

$$= -\frac{G_N M^2}{R} \cdot \left[\frac{3}{4} - \frac{1}{4} \cdot \frac{3}{5} \right]$$

$$= -\frac{3}{5}\frac{G_N M^2}{R} \quad \text{with} \quad M = \frac{N}{2} m(He^4)$$

The total energy is then

$$E = \frac{3}{5}\frac{\hbar^2}{2m_e}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}}{R^2} - \frac{3}{5} G_N \frac{N^2}{4} m(He^4) \cdot \frac{1}{R}$$

$$\frac{\partial E}{\partial R} = -2 \cdot \frac{3}{5}\frac{\hbar^2}{2m_e}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}}{R^3} + \frac{3}{5 \cdot 4} G_N N^2 m(He^4) \cdot \frac{1}{R^2}$$

this is minimized at

$$R = \underbrace{\frac{\hbar^2}{2m_e}}_{\text{ergs-cm}^2} \cdot \left(\frac{9\pi}{4}\right)^{2/3} \cdot \underbrace{\frac{4}{G_N m_{He^4}^2}}_{\frac{1}{\text{ergs cm}}} \cdot \frac{1}{N^{1/3}} = \text{cm} \checkmark$$

6.) Evaluate for a white dwarf with the mass of the sun

$$M = 2\times10^{33}\,g$$

$$N = \frac{M}{m_{He}/2} = 1.2\times10^{57}$$

$$\frac{\hbar^2}{2m_e} = 1.22\times10^{-27}\text{ erg cm}$$

$$G_N m_{He^4}^2 = 7.5\times10^{-55}\text{ erg cm}$$

$$R = 7.3\,\frac{\hbar^2}{2m_e}\,\frac{1}{G_N m_{He^4}^2}\cdot\frac{1}{N^{1/3}} = 1.1\times10^9\,cm$$

$$\sim 10^4\text{ km}$$

cf. $R_{earth} = 6.4\times10^3$ km

$$n = \frac{N}{\frac{4\pi}{3}R^3} = 2.2\times10^{29}/cm^3$$

$$n\cdot\frac{m(He^4)}{2} = 7.3\times10^5\,g/cm^3$$

$$p_F = 1.85\times10^{10}/cm$$

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\cdot p_F^2 = 2.09\times10^{-7}\,erg$$

$$\varepsilon_F/k_B = 1.6\times10^9\ {}^\circ K$$

then $\longrightarrow$

(i) $$\varepsilon_F = 2.1\times10^{-7}\,\text{erg} \times \frac{1\,\text{eV}}{1.6\times10^{-12}\,\text{erg}} = 1.3\times10^{5}\,\text{eV}$$

so $\varepsilon_F \gg$ atomic binding energies ($\approx$ few eV)

(ii) $$\varepsilon_F / k_B = 1.6\times10^{9}\,{}^\circ\text{K} \gg 10^{7}\,{}^\circ\text{K},$$ the temperature inside the star

(iii) $$m_e c^2 = 0.51\times10^{6}\,\text{eV}$$
$$(m_n - m_p)\,c^2 = 1.29\times10^{6}\,\text{eV}$$

so it is still acceptable to treat electrons as non relativistic and to ignore $e^- + p \rightleftarrows n + \nu$ ← $\mu = 0$

C.) Now assume $E = pc$ for electrons.

$$\varepsilon_F = \hbar c \cdot p_F = \hbar c\left(\frac{9\pi}{4}\frac{N}{R^3}\right)^{1/3} \sim \frac{1}{R}$$

so now gravitational and degeneracy energies scale with the same power and it is not possible to balance them.

$$E = \frac{2V}{8\pi^3}\int_0^{p_F} d^3p\; cp = \frac{3}{4} N \varepsilon_F$$

so degeneracy energy: $$E = \frac{3}{4}\hbar c\left(\frac{9\pi}{4}\right)^{1/3}\frac{N^{4/3}}{R}$$

gravitational energy: $$E_G = -\frac{3}{5}\,\frac{G_N m_{He}^2}{4}\,\frac{N^2}{R}$$

The coefficients are equal for

$$\frac{3}{4}\hbar c\left(\frac{9\pi}{4}\right)^{1/3} N^{4/3} = \frac{3}{5}\frac{G_N m_{He}^2}{4} N^2$$

$$N^{2/3} = 5\left(\frac{9\pi}{4}\right)^{1/3}\frac{\hbar c}{G_N m_{He^4}} = 4.0 \times 10^{38}$$

or

$$N = 8.0 \times 10^{57} = 7 \times M_\odot$$

Above this mass or number of He^4's, the star would collapse to a neutron star. Actually, the threshold for collapse is lower, since the star would be denser at the center in a more realistic model. Taking this into account, one finds

$$M_{\text{Chandrasekhar}} \sim 1.5\, M_\odot$$