Physics 212 — Problem Set #3

Solutions

1.) a.) In this problem, "thought" is a process that liberates ΔS per thought. Essentially, we decrease the entropy of some register or display by n bits and, since entropy always increases, we must release this entropy as heat. If this last release is done by transferring energy ΔQ from a hot reservoir to a cold reservoir

$$\Delta S = \frac{\Delta Q}{T_2} - \frac{\Delta Q}{T_1} \qquad T_1 > T_2$$

if $T_1 \gg T_2$ $\qquad \Delta S \approx \frac{\Delta Q}{T_2}$ $\quad$ or $\quad \Delta Q = T_2 \Delta S$

b.) Now assume that the maximum rate at which we can radiate energy at T_2 is

$$\frac{dQ}{dt} \leq C T_2^3$$

Let H = "thoughts" = (S liberated)$/\Delta S$

$$\frac{dH}{dt} \leq \frac{C}{\Delta S} T_2^2$$

[Why $\frac{dQ}{dt} < CT^3$ rather that $\frac{dQ}{dt} < CT^4$ as for a black body? Dyson proves in his article that

$$\frac{dQ}{dt} < 2.8 \frac{Ne^2}{m_e \hbar^2 c^3} (k_B T)^3$$

is the absolute upper limit for the power that can be radiated by N electrons at temperature T.

c.) Assuming a matter dominated universe with zero cosmological constant,

$$R \sim t^{2/3}$$

and the temperature of the cosmic microwave background goes as

$$\Theta \sim R^{-1} \sim t^{-2/3}$$

If we choose $T_2 = \frac{A}{t^{2/3}}$ "ecologically efficient being"

$$\frac{dH}{dt} \leq \frac{c}{\Delta S} \frac{A^2}{t^{4/3}}$$

$$H = \int_{t_0}^{\infty} dt \frac{dH}{dt} \leq 3 \frac{c}{\Delta S} A^2 \frac{1}{t_0^{1/3}}$$

which is finite.

[Sethna uses $T_2 \sim \frac{1}{t}$; this also gives a finite result.]

d.) Dyson proposes $T_2 = B/t^{3/8}$ "profligate beings"

then

$$H = \int_{t_0}^{\infty} dt \frac{dH}{dt} \leq \frac{c}{\Delta S} B^2 \int \frac{dt}{t^{3/4}}$$

~~which~~ is infinite

while the total heat released is

$$Q = \int dt \; C \left(\frac{B}{t^{3/8}}\right)^3 = C B^3 \int_{t_0}^{\infty} \frac{dt}{t^{9/8}}$$

$$= 8 C B^3 \frac{1}{t_0^{1/8}}$$

which is finite. Actually

$$T_2 = B/t^{\alpha} \quad \text{for } \alpha \text{ in the range } \tfrac{1}{3} < \alpha < \tfrac{1}{2}$$

will work.

Unfortunately, our universe actually appears to have a nonzero cosmological constant. The expansion of the universe is already dominated by the cosmological constant. A de Sitter universe has a nonzero temperature $T \sim \sqrt{G_N \Lambda}$. For the measured value of Λ this gives $T \sim 10^{-33}\,\mathrm{eV}$. But still, it puts a limit on the total number of thoughts, since we cannot operate a heat engine below this temperature.

For further reading, see

F. J. Dyson Rev. Mod. Phys. <u>51</u> 447 (1979)

L. Krauss + G. Starkman, Ap. J. <u>531</u> 22 (200),
astro-ph/9902189

2.) a) Let the volume of the box be V in the compressed state, $2V$ in the expanded state. The work needed to compress the box at fixed T is

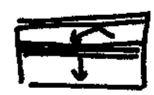
$$W = \int_V^{2V} \frac{dV}{V} \cdot T = T \log 2 \qquad \text{since } N=1$$

b.) To convert a bit 0 to a bit 1:

① Compress the top piston.

② Insert a partition at the bottom

③ Move both pistons upward at the same rate.



④ Lower the bottom piston

This is a reversible process that requires no net work.

c.) Removing a partition when the atom position is known increases the entropy of the system by

$$\Delta S = \Delta \log V = \log 2$$

[The information entropy increases by $\log_2 2 = 1$ bit.]

d.) The demon can extract $W = T \log 2$ by learning the position of an atom and transferring the information to one of its internal states. But, this state is known only to the demon, not to the outside world. As far as the outside world is concerned, the demon has reversibly transferred $\Delta S = \log 2$ from the tape to its own internal registers. An external observer would need to use up $W = T \log 2$ to restore the demon to its original state.

3.) a.) XOR is the rule

$$0 \oplus 0 = 0$$
$$0 \oplus 1 = 1$$
$$1 \oplus 0 = 1$$
$$1 \oplus 1 = 0$$

the original state has information entropy $\log 4 = 2 \log 2$
the final state has information entropy $\log 2 = \log 2$

so information $\log 2$ is destroyed, requiring work $\Delta W = T \log 2$

b.) Controlled-Not is the rule

$$0 \oslash 0 = (0,0)$$
$$0 \oslash 1 = (0,1)$$
$$1 \oslash 0 = (1,1)$$
$$1 \oslash 1 = (1,0)$$

$$AB \xrightarrow[\oslash]{} CD \xrightarrow[\oslash]{} EF \quad ?$$

$$(0,0) \rightarrow (0,0) \rightarrow (0,0)$$
$$(0,1) \rightarrow (0,1) \rightarrow (0,1)$$
$$(1,0) \rightarrow (1,1) \rightarrow (1,0)$$
$$(1,1) \rightarrow (1,0) \rightarrow (1,1)$$

so $(\text{controlled-Not})^2 = 1$ this must be reversible.

4.) a.) Area of the refrigerator:

$$4 \times 2m^2 + 2 \times 1\ m^2 = 10\ m^2$$

ΔT across the insulation $= 30°K$

$$\frac{\Delta T}{\Delta x} = 1000°K/m$$

thermal conductivity $D_T = 0.02\ W/m°K$

so the heat leak is

$$D_T \cdot \frac{\Delta T}{\Delta x} \cdot \text{Area} = 0.02\ W/m°K \cdot \frac{10^3\ °K}{m} \cdot 10\ m^2$$

$$= 200\ W \quad (\cong 3 \text{ light bulbs})$$

b.) The motor draws power $IV = 570.\ W$

The Carnot efficiency of the refrigerator is

$$\frac{\text{heat extracted at } T_2}{\text{net work}} = \frac{T_2}{T_1 - T_2} = \frac{270}{30} = 9$$

so we can cool the interior by 200 W using 22 W of power.

The motor will need to operate <u>4%</u> of the time

5.) a.) Model a rubber band by $\underline{N}$ links of length $\underline{d}$. The total change in position from one end to the other is $\underline{L}$. Let the orientation of the links be random, then

$$\langle L \rangle = 0$$

L measures the deviation from the rest position of the rubber band.

A rubber band at fixed L is in one of

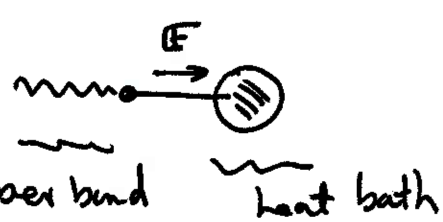
$$\binom{N}{l} = \frac{N!}{l!\,(N-l)!} \qquad \text{states}$$

with $$[\,l - (N-l)\,]\cdot d = L$$

or $$l = \frac{N + L/d}{2}$$

$$S = \log(\#\text{ of configs}) = \log\left(\frac{N!}{l!\,(N-l)!}\right)$$

b.) Now imagine that the band is maintained at a temperature T and a force $\mathbb{F}$ is applied



$$0 = \frac{d}{dL_1}\left(S_{band} + S_{bath}\right) = \frac{dS_{band}}{dL_1} + \underbrace{\frac{dE}{dL_1}}_{-F}\ \underbrace{\frac{dS_{bath}}{dE}}_{\frac{1}{T}}$$

so

$$F = T\,\frac{dS_{band}}{dL_1}$$

c.) For $N \gg \ell$

$$\begin{aligned} S_{band} &= \log N! - \log \ell! - \log (N-\ell)! \\ &= N\lg N - N - \ell \lg \ell + \ell - (N-\ell)\lg(N-\ell) \\ &\qquad + (N-\ell) \\ &= N\lg N - \ell \lg \ell - (N-\ell)\lg(N-\ell) \end{aligned}$$

$\ell = \frac{N}{2} + \frac{L_1}{2d}$ so

$$\begin{aligned} \log \ell &= \log\frac{N}{2} + \log\left(1 + \frac{L_1}{Nd}\right) \\ &= \log\frac{N}{2} + \frac{L_1}{Nd} - \frac{1}{2}\left(\frac{L_1}{Nd}\right)^2 + \cdots \end{aligned}$$

$$\begin{aligned} \ell \log \ell &= N/2 \log N/2 + \frac{L_1}{2d}\log N/2 + \frac{L_1}{2d} - \frac{L_1^2}{4Nd^2} + \frac{L_1^2}{2Nd^2} + \cdots \\ &= N/2 \log N/2 + \frac{L_1}{2d}\log N/2 + \frac{L_1}{2d} + \frac{L_1^2}{4Nd^2} + \cdots \end{aligned}$$

$$(N-\ell) = N/2 - \frac{L_1}{2d}$$

so $(N-\ell)\lg(N-\ell) = N/2 \lg N/2 - \frac{L_1}{2d}\log\frac{N}{2} - \frac{L_1}{2d} + \frac{L_1^2}{4Nd^2} + \cdots$

in all

$$S_{band} = N \log N - N \log \frac{N}{2} - \frac{L^2}{2Nd^2} + \cdots$$

$$S_{band} = N \log 2 - \frac{L^2}{2Nd^2}$$

$$\mathbb{F} = -\frac{T}{Nd^2} L$$

d.) Increase T for fixed $\mathbb{F}$

$\Rightarrow$ L decreases, band contracts

Here is a "thermodynamic" way to see this.

$$dS = \frac{dE}{T} - \frac{\mathbb{F}}{T} dL$$

Define an appropriate Gibbs free energy

$$G = E - TS - \mathbb{F} L$$

$$dE = TdS + \mathbb{F}\, dL$$

$$dG = -S\,dT - L\,d\mathbb{F}$$

Then there is a Maxwell relation

$$\frac{\partial^2 G}{\partial T \partial \mathbb{F}} = -\left.\frac{\partial S}{\partial \mathbb{F}}\right|_T = -\left.\frac{\partial L}{\partial T}\right|_{\mathbb{F}}$$

so $$\left.\frac{\partial L}{\partial T}\right|_{\mathbb{F}} = \left.\frac{\partial S}{\partial \mathbb{F}}\right|_{T} < 0 . \qquad !$$

e.) When stretched, a rubber band heats up.
The entropy of stretching decreases, so entropy must go into molecular vibration.

The Maxwell relation following from E on p. 9 is

$$\left(\frac{\partial T}{\partial L}\right)_S = \left(\frac{\partial \mathbb{F}}{\partial S}\right)_L$$

At fixed L heating increases $\mathbb{F}$ so $\left(\frac{\partial \mathbb{F}}{\partial S}\right)_L > 0$

then

$$\left(\frac{\partial T}{\partial L}\right)_S > 0$$

When the rubber band relaxes, it cools (just conversely)

For an ideal gas, the system does work as it expands

$$dE = -PdV$$

For a rubber band, work is done on the system as it expands

$$dE = +\mathbb{F}dL$$

So it makes sense that an ideal gas <u>cools</u> but a rubber band <u>heats</u> when it is stretched.