Physics 212 – Problem Set 1
Solutions

1.) Uniform distribution: $\rho(x) = 1 \qquad 0<x<1$

Exponential distribution: $\rho(t) = \frac{1}{\tau}\, e^{-t/\tau} \qquad 0<t<\infty$

Gaussian distribution $\rho(v) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-v^2/2\sigma^2} \qquad -\infty<v<\infty$

a.) Uniform:

$$P(0.7 < x < 0.75) = \int_{0.7}^{0.75} dx\, \rho(x) = 0.05$$

Exponential:

$$P(t > 2\tau) = \int_{2\tau}^{\infty} dt\, \rho(t) = e^{-t/\tau}\Big|_{t=2\tau} = e^{-2} = 0.135$$

Gaussian:

$$P(v>2\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{2\sigma}^{\infty} dv\, e^{-v^2/2\sigma^2} = \int_{2}^{\infty} \frac{dv'}{\sqrt{2\pi}}\, e^{-v'^2/2} \approx 0.023$$

b.) Normalization, mean, standard deviation

Uniform:

$$\int_0^1 dx\, \rho(x) = \int_0^1 dx\, 1 = 1 \quad \checkmark$$

$$\langle x \rangle = \int_0^1 dx\, x = \frac{1}{2}$$

$$\langle x^2 \rangle = \int_0^1 dx\, x^2 = \frac{1}{3}$$

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma = \frac{1}{\sqrt{12}}$$ (It is worth memorizing this result.)

Exponential:

$$\int dt\, \rho(t) = \int_0^\infty \frac{dt}{\tau} e^{-t/\tau} = 1 \quad \checkmark$$

$$\langle t \rangle = \int_0^\infty \frac{dt}{\tau}\, t\, e^{-t/\tau} = \tau$$

$$\langle t^2 \rangle = \int_0^\infty \frac{dt}{\tau}\, t^2\, e^{-t/\tau} = 2\tau^2$$

$$\sigma^2 = 2\tau^2 - \tau^2 = \tau^2 \qquad \sigma = \tau$$

Gaussian: $\int dv\, \rho(v) = \int_{-\infty}^{\infty} \frac{dv}{\sqrt{2\pi\sigma^2}}\, e^{-v^2/2\sigma^2} = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{\pi}}\, e^{-y^2} = 1$ ✓

$$\langle v\rangle = \int dv \frac{1}{\sqrt{2\pi\sigma^2}}\, v\, e^{-v^2/2\sigma^2} = 0 \quad \text{by symmetry}$$

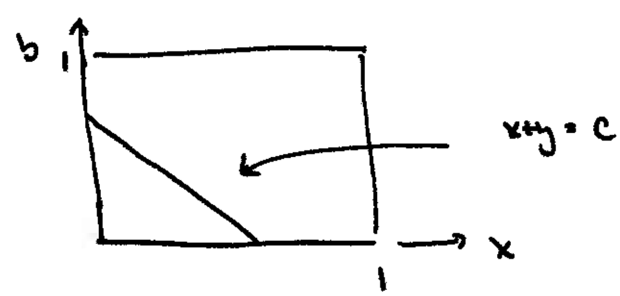
$$\langle v^2\rangle = \int dv \frac{1}{\sqrt{2\pi\sigma^2}}\, v^2\, e^{-v^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(-\frac{d}{d\lambda}\right)\left(\int_{-\infty}^{\infty} dv\, e^{-\lambda v^2}\right)\Big|_{\lambda=\frac{1}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi\sigma^2}} \left(-\frac{d}{d\lambda}\right)\sqrt{\frac{\pi}{\lambda}}\,\Big|_{\lambda=\frac{1}{2\sigma^2}} = \frac{1}{2}\frac{1}{\lambda}\Big|_{\lambda=\frac{1}{2\sigma^2}} = \sigma^2$$

so $\langle v^2\rangle - \langle v\rangle^2 = \sigma^2$ appropriately

c.) If x and y are uniformly distributed, the probability of a given value of x+y is the probability of lying on a diagonal of the unit square:



Then $P(z) = \langle x+y\rangle$ has a maximum at $z=1$, corresponding to the diagonal of the square.

$$P(z) = \begin{cases} z & z<1 \\ 2-z & z>1 \end{cases} \qquad z = x+y$$

has the correct shape and is normalized:

$$\int_0^2 dz\; P(z) = \tfrac{1}{2} + \tfrac{1}{2} = 1 \quad ✓$$

In general, the distribution of $z = x+y$ is given by

$$\int dz\, P(z) = \int dz \int dx\; \rho(x) \int dy\; \rho(y)\; \delta(z-(x+y))$$

Proof: integrating over z gives an identity. The right-hand side evaluates to

$$= \int dz \int dx\; \rho(x)\; \rho(y)\Big|_{y=z-x}$$

or

$$P(z) = \int dx\; \rho(x)\; \rho(z-x)$$

For a uniform distribution:

$$z < 1 \qquad P(z) = \int_0^z dx\; 1\cdot 1 = z$$

$$1 < z < 2 \qquad P(z) = \int_{z-1}^{1} dx\; 1\cdot 1 = 2-z$$

d) For a Maxwellian distribution

$$\rho(v_x, v_y, v_z) = \frac{1}{\sqrt{2\pi kT/M}}\, e^{-Mv_x^2/2kT} \cdot \frac{1}{\sqrt{2\pi kT/M}}\, e^{-Mv_y^2/2kT}$$

$$\cdot \frac{1}{\sqrt{2\pi kT/M}}\, e^{-Mv_z^2/2kT}$$

$$\langle V_x^2 \rangle = (\sigma^2 \text{ of 1-d distribution}) = \frac{kT}{M}$$

similarly $\left\langle \frac{MV_x^2}{2} \right\rangle = \left\langle \frac{MV_y^2}{2} \right\rangle = \left\langle \frac{MV_z^2}{2} \right\rangle = \frac{kT}{2}$

To work out $P(v)$ write the distribution as

$$\int dv_x\, dv_y\, dv_z \quad \frac{1}{(2\pi kT/M)^{3/2}}\, e^{-Mv^2/2kT}$$

and $d^3v = dv\, v^2 \cdot 4\pi$

$$= \int dv \quad v^2 \frac{4\pi}{(2\pi)^{3/2}} \frac{1}{(kT/M)^{3/2}}\, e^{-Mv^2/2kT}$$

$\sigma = \frac{kT}{M}$
$$= \int dv \sqrt{\frac{2}{\pi}}\, \frac{v^2}{\sigma^3}\, e^{-v^2/2\sigma^2}$$

$$= \int dv \; P(v) \quad \text{so} \quad P(v) = \sqrt{\frac{2}{\pi}}\, \frac{v^2}{\sigma^3}\, e^{-v^2/2\sigma^2}$$

2.) Buses appear exactly at $t = 0, 5, 10, \ldots$
Cars appear with a probability distribution

$$\frac{dt}{\tau} \qquad \tau = 5$$

a.) In $\Delta t = 60$, 12 buses pass. The mean number of cars passing is

$$\int_0^{60} \frac{dt}{\tau} = \frac{60}{5} = 12$$

b.) In 10 minutes, exactly 2 buses will pass:

$$P_{bus}(0) = 0 \qquad P_{bus}(1) = 0 \qquad P_{bus}(2) = 1 \qquad P_{bus}(3) = 0$$

etc.

For cars, this is a trickier problem. In any small interval of time, the probability of seeing one car is $\Delta t/\tau$. But in a finite interval, we can see 2, 3, ... cars.

Let's first compute the probability of seeing no cars. This is $P_0(t)$. For $t=0$, $P_0(0) = 1$. For a small interval $P_0(t) \approx 1 - t/\tau$. Actually, if $P_0(t)$ is the probability of seeing no cars after time t, $P_0(t+\Delta t) = P_0(t)\left(1 - \frac{\Delta t}{\tau}\right)$

Then $P_0(t)$ obeys:

$$\frac{d}{dt} P_0(t) = -\frac{1}{\tau} P_0(t)$$

The solution is $\quad P_0(t) = e^{-t/\tau}$.

Now let $P_1(t)$ be the probability that we see exactly one car in the interval. This obeys:

$$\frac{d}{dt} P_1(t) = \underbrace{\frac{1}{\tau} P_0(t)}_{\text{a car appears}} - \underbrace{\frac{1}{\tau} P_1(t)}_{\text{a second car appears.}}$$

The solution, with $P_1(t) = 0$ at $t=0$ is

$$P_1(t) = \frac{t}{\tau} e^{-t/\tau}$$

similarly $P_2(t)$ satisfies $\frac{d}{dt}P_2(t) = \frac{1}{\tau}P_1(t) - \frac{1}{\tau}P_2(t)$

and $P_2(t)=0$ at $t=0$, so $P_2(t) = \frac{1}{2}\left(\frac{t}{\tau}\right)^2 e^{-t/\tau}$

in general

$$\frac{d}{dt}P_n(t) = \frac{1}{\tau}P_{n-1}(t) - \frac{1}{\tau}P_n(t)$$

$$P_n(t) = \frac{1}{n!}\left(\frac{t}{\tau}\right)^n e^{-t/\tau}$$

A check is

$$\sum_{n=0}^{\infty} P_n(t) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{\tau}\right)^n e^{-t/\tau} = e^{t/\tau}e^{-t/\tau} = 1 \checkmark$$

This has the form of a Poisson distribution $P_n = \frac{\lambda^n}{n!}e^{-\lambda}$

c.) If a bus passes at $t=0$, the next bus will pass at $t=5$. So

$$P_{next,\,bus} = \delta(t-5)$$

If a car passes at $t=0$, the probability that there will be no further car at $t<t$ is

$$P_0(t) = e^{-t/\tau}$$

Then the probability that the next car will pass at t is given by

$$\int dt\, P_{next\;car}(t) = \int \frac{dt}{\tau} P_0(t) \Rightarrow P_{next\,car}(t) = \frac{1}{\tau}e^{-t/\tau}$$

In these distributions

$P_{\text{next-bus}}$: $\langle t \rangle = 5$

$P_{\text{next-car}}$: $\langle t \rangle = \tau = 5$ mean of an exponential distribution

d.) For an observer arriving at a random time, the interval to the next bus is a uniform random value between 0 and 5

so
$$\langle t \rangle = 2.5$$

The interval to the next car is computed exactly as in part (c)
$$\langle t \rangle = 5$$

Also, the interval to the previous car obeys $\langle t \rangle = 5$

e.) We can interpret the car answer in part c as the probability of a gap between cars of length Δ

$$P_{gap}(\Delta) = \frac{1}{\tau} e^{-\Delta/\tau}$$

If an observer arrives at a random time, the probability that he arrives in a gap of length Δ is proportional to Δ.

so
$$P_{obs}(\Delta) = C \frac{\Delta}{\tau} e^{-\Delta/\tau}$$

The normalized distribution is

$$P_{obs}(\Delta) = \frac{\Delta}{\tau^2} e^{-\Delta/\tau}$$

In this distribution

$$\langle \Delta \rangle = \int_0^\infty d\Delta \; \Delta \; \frac{\Delta}{\tau^2} \, e^{-\Delta/\tau}$$

$$= \tau \int dx \; x^2 e^{-x} = 2\tau$$

so $\langle \Delta \rangle = 2\tau = 10.$

3.) The mean free path for a photon is $\ell = 5 \times 10^{-5}$ m
To reach a radius R takes N steps, with

$$R = N^2 \ell$$

For $R = 5 \times 10^8$ m $\qquad N = 10^{26}$

At the speed of light, one step takes

$$t = 5 \times 10^{-5} \text{ m} / 3 \times 10^8 \text{ m/sec} = 1.7 \times 10^{-13} \text{ sec}$$

so the time to reach R is

$$1.7 \times 10^{+13} \text{ sec} = 5.3 \times 10^5 \text{ yrs.}$$

4.) a.) $\log n! = \sum_1^n \log n$

$$\simeq \int_1^n dn \, \log n = n \log n - n \Big|_1^n$$

$$\simeq n \log - n$$

b.) More exactly.

$$\sum_1^n \log n = \int_1^n dn \, \log n + \frac{1}{2}(\log n + \log 1)$$

$$+ \frac{1}{12}\left(\frac{1}{n} - 1\right) + \left(-\frac{1}{30 \cdot 24}\right)\left(\frac{2}{n^3} - 2\right) + \cdots$$

$$= (n \log n - n)\Big|_1^n + \frac{1}{2}\log n + O\left(\frac{1}{n}\right)$$

$$+ (\text{miscellaneous terms of order } 1)$$

$$= (n + \tfrac{1}{2}) \log n - n + O(1)$$

This method is not powerful enough to obtain the $O(1)$ term.

c.) $$\Gamma(z) = \int_0^\infty dx \, x^{z-1} e^{-x}$$

$$\Gamma(1) = \int_0^\infty dx \, e^{-x} = 1$$

$$\Gamma(z+1) = \int_0^\infty dx \, x^z e^{-x} = x^z(-e^{-x})\Big|_0^\infty - \int dx \, z x^{z-1}(-e^{-x})$$

$$= \int_0^\infty dx \, z x^{z-1} e^{-x} = z \, \Gamma(z)$$

so

$$\Gamma(2) = 1 \qquad \Gamma(3) = 2\cdot 1 \qquad \Gamma(4) = 3\cdot 2\cdot 1$$

$$\Gamma(n+1) = n!$$

d.)

$$n! = \int_0^\infty dx\; x^n e^{-x} = \int_0^\infty dx\; \exp[-x + n\log x]$$

for $n \gg 1$ this is a sharply peaked function. Its maximum is at

$$0 = \frac{d}{dx}[-x + n\log x] = -1 + \frac{n}{x} \Rightarrow n = x$$

so expand: $\quad x = n + \xi$

$$\log x = \log n\left(1 + \frac{\xi}{n}\right) = \log n + \frac{\xi}{n} - \frac{1}{2}\frac{\xi^2}{n^2} + \frac{1}{3}\frac{\xi^3}{n^3} - \frac{1}{4}\frac{\xi^4}{n^4} + \cdots$$

$\exp[-x + n\log x]$

$$= \exp\left[-n - \cancel{\xi} + n\log n + \cancel{\xi} - \frac{1}{2}\frac{\xi^2}{n} + \frac{1}{3}\frac{\xi^3}{n^2} - \frac{1}{4}\frac{\xi^4}{n^3} + \dots\right]$$

$$= \exp[n\log n - n]\; e^{-\frac{1}{2}\xi^2/n}\left(1 + \frac{1}{3}\frac{\xi^3}{n^2} - \frac{1}{4}\frac{\xi^4}{n^3} + \dots\right)$$

Ignore ( ) and integrate over ξ. This gives

$$n! = \exp[n\log n - n]\cdot\sqrt{2\pi n}$$

or $\quad \log n! = (n+\tfrac{1}{2})\log n - n + \log\sqrt{2\pi}$

Under the integral

$$\int d\xi \; e^{-\xi^2/2n}$$

the ξ^3 term integrates to 0. But

$$(\;) = 1 + \frac{1}{3}\frac{\xi^3}{n^2} - \frac{1}{4}\frac{\xi^4}{n^3} + \frac{1}{2\cdot 9}\frac{\xi^6}{n^4} + \cdots$$

$$\frac{1}{\sqrt{2\pi n}}\int_0^\infty d\xi \; e^{-\xi^2/2n} (\;) = 1 - \frac{1}{4}\cdot\frac{3n^2}{n^3} + \frac{1}{2\cdot 9}\, 5\cdot 3\,\frac{n^3}{n^4} + O\left(\frac{1}{n^2}\right)$$

$$= 1 - \frac{3}{4}\frac{1}{n} + \frac{5}{6}\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{1}{12}\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

so we have:

$$n! = n\log n - n \qquad (A)$$

$$(n+\tfrac{1}{2})\log n - n \qquad (B)$$

$$(n+\tfrac{1}{2})\log n - n + \log\sqrt{2\pi} \qquad (C)$$

$$(n+\tfrac{1}{2})\log n - n + \log\sqrt{2\pi} + \frac{1}{12}\frac{1}{n} \qquad (D)$$

e.) table of values of $\log n!$ and approximations:

n	log n!	A	B	C	D
3	1.7917595	0.29583687	0.84514301	2.6830201	2.7107979
10	15.104413	13.025851	14.177143	16.015021	16.023354
30	74.658236	72.035921	73.73652	75.574397	75.577175
100	363.73938	360.51702	362.8196	364.65748	364.65831
300	1414.9058	1411.1347	1413.9866	1415.8245	1415.8248
1000	5912.1282	5907.7553	5911.2092	5913.047	5913.0471
3000	21024.025	21019.103	21023.106	21024.944	21024.944
10000	82108.928	82103.404	82108.009	82109.847	82109.847
30000	279274.65	279268.58	279273.73	279275.57	279275.57
100000	1051299.2	1051292.5	1051298.3	1051300.1	1051300.1