Physics 212 – Problem Set # 1

(due Thursday, April 8)

Many of the problems in this course will be taken from the textbook "Statistical Mechanics: Entropy, Order Parameters, and Complexity", by James Sethna. Sethna has obviously put a large amount of effort into crafting these problems. We can take advantage of this.

1. Sethna, problem 1.2.
2. Sethna, problem 1.3.
3. Sethna, problem 2.2.
4. Give a few proofs of Stirling's formula for $n!$:
   (a) Write $\log n! = \sum_1^n \log n$. Approximate the sum by an integral and show that $\log n! \sim n \log n - n$.

   (b) There is a more exact way to approximate a series by an integral. This is done with the *Euler-Maclaurin summation formula*

$$\sum_{n_1}^{n_2} F(n) = \int_{n_1}^{n_2} dn\, F(n) + \frac{1}{2}(F(n_1)+F(n_2)) + \frac{B_2}{2!}[F'(n_2)-F'(n_1)] + \frac{B_4}{4!}[F^{(3)}(n_2)-F^{(3)}(n_1)] + \cdots \tag{1}$$

   where $F^{(m)}(x)$ is the mth derivative of $F(x)$ and B_n are the Bernoulli numbers: $B_2 = 1/6$, $B_4 = -1/30$, etc. Use this formula to obtain a more exact value of $\log n!$. But wasn't there supposed to be a π in the formula? What happened to it?

   (c) The Gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} \tag{2}$$

   Show that $\Gamma(1) = 1$. By integration by parts, show that $\Gamma(z+1) = z\Gamma(z)$. Conclude that $\Gamma(n+1) = n!$.

   (d) Show that, for very large n, the integrand in (2) has its maximum at $x = n$. Write $x = n + \xi$, expand the integrand in powers of ξ, and integrate. Show that

$$\log n! = (n + \frac{1}{2}) \log n - n + \log \sqrt{2\pi} + O(\frac{1}{n}) \tag{3}$$

   and derive the next term in the series.

   (e) Compare these approximations to the exact results for $\log n!$ for $n = 3$, 10, 30, 100, 300, 1000.