Apr. 29

Magnetism (continued)

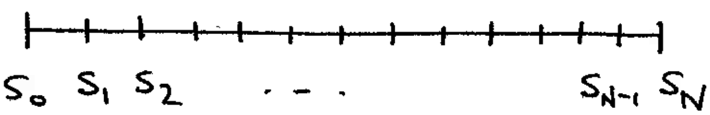
In the previous lecture, we worked out a complete picture of the phases of the Ising model using the approximation of mean field theory. I carried out this analysis for a 3 dimensional cubic lattice, but in fact the analysis generalizes easily and gives the same qualitative results in any dimensionality. In particular, mean field theory predicts a critical point at

$$T_c = 2dJ$$

in d dimensions, with spontaneous magnetization for $T < T_c$ and various sorts of singularities in thermodynamic functions in the neighborhood of this point.

I will now test these conclusions by deriving some exact results for the Ising model. To begin, I will give the exact solution to the Ising model in 1 dimension. This problem was first solved in Ising's Ph.D. thesis.

We can view the 1 dimensional Ising model as a chain of spins along a line, with N nearest-neighbor interactions,



I will first compute the partition function of this system with the boundary spins s_0 and s_N fixed. The partition function is

$$Z_N(s_0, s_N) = \sum_{s_1, s_2 \cdots s_{N-1} = \pm 1} e^{\beta J \sum_0^{N-1} s_i s_{i+1}} \, e^{\beta \mu h \sum_0^{N-1} s_i}$$

Given this partition function, we can easily compute the partition functions with other boundary conditions. For example, the partition function with periodic boundary conditions is

$$Z_{N,\text{per.}} = \sum_{s_0=\pm 1} Z_N(s_0, s_0)$$

I will first compute $Z_N(s_0, s_N)$ in zero magnetic field, $h = 0$. This can be done in an organized way as follows: Imagine that we have computed $Z_M(s_0, s_M)$ for a fixed length M. The partition function for the system with one more spin is given by

$$Z_{M+1}(s_0, s_{M+1}) = \sum_{s_M=\pm 1} Z_M(s_0, s_M)\, e^{\beta J s_M s_{M+1}}$$

We can represent the two possible boundary conditions at each end of the chain by 2-component vectors,

$$\langle s_0| = (1\ 0) \text{ for } \uparrow \qquad (0\ 1) \text{ for } \downarrow$$

$$|s_N\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } \uparrow \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for } \downarrow$$

Then we can view $Z_M(s_0, s_M)$ for the two possible values of s_M as a 2×2 matrix

$$Z_M = \begin{pmatrix} Z_M(\uparrow,\uparrow) & Z_M(\uparrow,\downarrow) \\ Z_M(\downarrow,\uparrow) & Z_M(\downarrow,\downarrow) \end{pmatrix}$$

We can also represent the second factor in the expression for Z_{M+1} as a matrix

$$(T)_{s_M s_{M+1}} = e^{\beta J s_M s_{M+1}} = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix}$$

and view the sum involved in constructing Z_{M+1} as matrix multiplication,

$$Z_{M+1} = Z_M \cdot T$$

The matrix T is called the *transfer matrix.* We can now compute Z_N by interating this process,

$$Z_N(s_0, s_N) = \langle s_0 | \, T^N \, | s_N \rangle$$

The partition function with periodic boundary conditions is given by

$$Z_{N,\text{per}} = \operatorname{tr} T^N$$

In the case at hand, T is self-adjoint, so we can compute these quantities explicitly by diagonalizing T. There are two eigenvectors, with the eigenvalues

$$|v_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_0 = e^{\beta J} + e^{-\beta J} = 2\cosh \beta J$$

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda_1 = e^{\beta J} - e^{-\beta J} = 2\sinh \beta J$$

Then the partition function with periodic boundary conditions is

$$Z_{N,\text{per}} = \lambda_0^N + \lambda_1^N$$

Now, $\lambda_0 > \lambda_1$ and N is 10^{20} or so, so λ_1^N/λ_0^N is a *very* small number. Then

$$Z_{N,per} = \lambda_0^N = (2\cosh\beta J)^N$$

From this, it follows that the free energy of the Ising model is

$$F = -T\log Z_{N,per} = -NT\log(2\cosh\beta J)$$

and the thermal expectation value of the energy is

$$E = -\frac{\partial}{\partial\beta}\log Z_N = -JN\tanh\beta J$$

As $\beta \to 0$, high temperature,

$$E \to -\beta J^2 N \to 0$$

As $\beta \to \infty$, low temperature,

$$E \to -JN$$

and all spins are aligned.

Please note that F and E are analytic functions of β. There is no sign of a critical point in this solution.

It is not much more difficult to solve the 1 dimensional Ising model for nonzero magnetic field. From that solution, we can compute the magnetic susceptibility and the magnetization as a function of h. At nonzero field, the factor that must be added to add one further vertex to the chain of spins is

$$e^{\beta J s_M s_{M+1}} \cdot e^{\beta\mu h s_{M+1}}$$

This gives the transfer matrix

$$T = \begin{pmatrix} e^{\beta J} e^{\beta\mu h} & e^{-\beta J} e^{-\beta\mu h} \\ e^{\beta J} e^{\beta\mu h} & e^{-\beta J} e^{-\beta\mu h} \end{pmatrix}$$

It is most convenient to write this as

$$T = \begin{pmatrix} e^{-\beta\mu h/2} & 0 \\ 0 & e^{+\beta\mu h/2} \end{pmatrix} \begin{pmatrix} e^{\beta J} e^{\beta\mu h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} e^{-\beta\mu h} \end{pmatrix} \begin{pmatrix} e^{+\beta\mu h/2} & 0 \\ 0 & e^{-\beta\mu h/2} \end{pmatrix}$$

$$= A \quad T' \quad A^{-1}$$

with T' a self-adjoint matrix. Now

$$Z_{N,\text{per.}} = \operatorname{tr} T^N = \operatorname{tr} (T')^N = (\lambda_0')^N$$

if λ_0' is the largest eigenvalue of T'. It is not hard to diagonalize T'. Its eigenvalues are

$$\lambda_\pm = e^{\beta J} \cosh \beta\mu h \pm \left(e^{2\beta J} \sinh^2 \beta\mu h + e^{-2\beta J} \right)^{1/2}$$

The larger eigenvalue is the one with $\pm \to +$. Thus,

$$Z_{N,\mu h} = \left[e^{\beta J} \cosh \beta\mu h + \left(e^{2\beta J} \sinh^2 \beta\mu h + e^{-2\beta J} \right)^{1/2} \right]^N$$

From this, we can compute the magnetization

$$M = \frac{1}{\beta\mu} \frac{\partial}{\partial h} \log Z_N$$

$$= N \frac{e^{\beta J} \sinh \beta\mu h + \dfrac{e^{2\beta J} \sinh \beta\mu h \cosh \beta\mu h}{\left(e^{2\beta J} \sinh^2 \beta\mu h + e^{-2\beta J} \right)^{1/2}}}{\left[e^{\beta J} \cosh \beta\mu h + \left(e^{2\beta J} \sinh^2 \beta\mu h + e^{-2\beta J} \right)^{1/2} \right]}$$

Note that this expression goes smoothly to zero as $h \to 0$. Thus, in the 1 dimensional Ising model, $M(h)$ has no singularity or discontinuity as a function of h and $M = 0$ at zero field at all finite temperatures. Similarly, the zero-field susceptibility is

$$\chi(h=0) = \left.\frac{\partial M}{\partial h}\right|_{h=0} = N\beta\mu \frac{e^{\beta J} + \dfrac{e^{2\beta J}}{e^{-\beta J}}}{\left[e^{\beta J} + e^{-\beta J} \right]}$$

so that

$$\chi\Big|_{h=0} = \frac{N\mu}{T} \cdot e^{2\beta J} \left(\frac{e^{-\beta J} + e^{\beta J}}{e^{\beta J} + e^{-\beta J}} \right)$$

or, finally,

$$\chi\Big|_{h=0} = \frac{N\mu}{T} \cdot e^{2\beta J}$$

This expression is a simple product of a Curie-Weiss law and an analytic enhancement factor that becomes large as $\beta \to \infty$ or $T \to 0$. The zero-field susceptibility has no singularity at any $T > 0$.

Our explicit solution to the 1 dimensional Ising model allows us to compute several more interesting quantities. In the following discussion, I will set $h = 0$.

First, consider a problem in which we put $s_0 = +1$ on the left-hand boundary. Then s_1 will tend to have spin up also, and similarly for s_2. The influence decreases as we go farther away from the boundary, and we know that, for $h = 0$,

$$\langle s_n \rangle \to 0 \quad \text{as} \quad n \to \infty$$

It is interesting to compute the explicit dependence of $\langle s_n \rangle$ on n.

For this computation, let

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the operator on the 2-state Hilbert space that measures the value of the spin s_i. To evaluate $\langle s_n \rangle$, we put this operator on the site n,

$$\langle s_n \rangle = \frac{\langle s_0 | T^n \, S \, T^{N-n} | s_N \rangle}{\langle s_0 | T^N | s_N \rangle}$$

In this problem, n is finite but N is huge, so T^N projects onto the leading eigenvector

$$\langle s_n \rangle = \frac{\langle s_0 | T^n S | v_0 \rangle \lambda_0^{N-n} \langle v_0 | s_N \rangle}{\langle s_0 | v_0 \rangle \lambda_0^N \langle v_0 | s_N \rangle}$$

We can rewrite S as

$$S = \tfrac{1}{2} (1\ 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \tfrac{1}{2} (1\ -1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= |v_0\rangle \langle v_1| + |v_1\rangle \langle v_0|$$

This formula has an interesting explanation. The Ising model has a symmetry

$$\mathcal{P} : \quad s_i \to -s_i$$

This symmetry is implemented by the parity transformation

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $h = 0$, P commutes with T. Then the eigenvectors of T have definite parity,

$$\mathcal{P}|v_0\rangle = +|v_0\rangle \qquad \mathcal{P}|v_1\rangle = -|v_1\rangle$$

Parity reverses the spin operator (and vice versa)

$$\mathcal{P}S = -S\mathcal{P}$$

so it must be that S flips the state between the even parity state $|v_0\rangle$ and the odd parity state $|v_1\rangle$. If we insert this representation of S into the expression for $\langle s_n\rangle$, we find

$$\langle s_n\rangle = \frac{\langle s_0|T^n|v_1\rangle \lambda_0^{N-n}\langle v_0|s_N\rangle}{\langle s_0|v_0\rangle \lambda_0^N \langle v_0|s_N\rangle} = \frac{\langle s_0|v_1\rangle}{\langle s_0|v_0\rangle}\frac{\lambda_1^n}{\lambda_0^n}$$

Then, finally,

$$\langle s_n \rangle = \frac{\langle s_0 | v_1 \rangle}{\langle s_0 | v_0 \rangle} \cdot \left(\frac{\lambda_1}{\lambda_0} \right)^n = (\text{const}) \cdot (\tanh \beta J)^n$$

The expectation value of s_n goes *exponentially* to zero as we move away from the boundary.

By a similar method, we can compute the *spin-spin correlation function*

$$\langle s_n s_m \rangle$$

I will assume that m, n are both very large (and, thus, well away from the boundaries) with $|m - n|$ finite. For definiteness, take $n \leq m$. Then

$$\langle s_n s_m \rangle = \frac{\langle s_0 | T^n S T^{m-n} S T^{N-m} | s_N \rangle}{\langle s_0 | T^N | s_N \rangle}$$

If n and $N - m$ are large, T^n and T^{N-m} are dominated by λ_0 and v_0. Thus

$$\langle s_n s_m \rangle = \frac{\langle s_0 | v_0 \rangle \lambda_0^n \langle v_0 | S T^{m-n} S | v_0 \rangle \lambda_0^{N-m} \langle v_0 | s_N \rangle}{\langle s_0 | v_0 \rangle \lambda_0^N \langle v_0 | s_N \rangle}$$

Since S flips the parity of the state from $|v_0\rangle$ to $|v_1\rangle$,

$$\langle s_n s_m \rangle = \frac{\langle v_1 | T^{m-n} | v_1 \rangle}{\lambda_0^{m-n}} = \left(\frac{\lambda_1}{\lambda_0} \right)^{m-n} = (\tanh \beta J)^{m-n}$$

So we find an *exponential* decay of the correlation

$$\langle s_n s_m \rangle = \exp \left[- |m-n| \log \coth \beta J \right]$$

This formula is conventionally written

$$\langle s_n s_m \rangle = (\text{const}) \cdot \exp\left[- \frac{|m-n|}{\xi(T)} \right]$$

where $\xi(T)$ is called the *correlation length.* For this model

$$\xi(T) = \frac{1}{\log(\coth \beta J)}$$

As $\beta \to 0$, $T \to \infty$,

$$\xi(T) \sim \frac{1}{\log T/J} \to 0$$

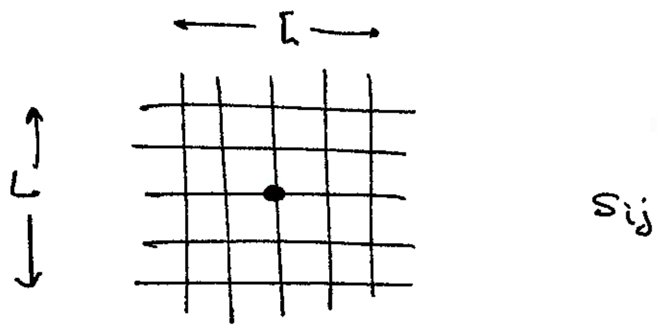
Since

$$\coth \beta J = \frac{1 + e^{-2\beta J}}{1 - e^{-2\beta J}}$$

we see that, as $\beta \to \infty$, $T \to 0$,

$$\xi(T) \sim \frac{e^{2\beta J}}{2} \to \infty$$

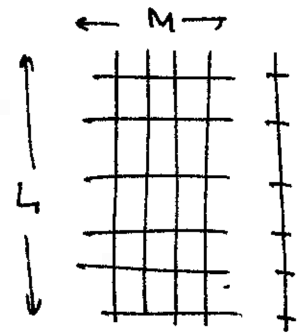
The formalism of the transfer matrix generalizes to lattice spin systems in higher dimension. Consider, for example, the 2 dimensional Ising model. We can consider this system to lives on an $L \times L$ lattice, with $L^2 = N$. The spins are labelled with two coordinates, s_{ij}.



The partition function is

$$Z = \sum_{\{s_{ij}\}} e^{\beta J \sum_{ij} (s_{ij} s_{i+1j} + s_{ij} s_{ij+1})}$$

In analogy to our strategy in 1 dimension, we can compute the partition function by first computing this function for a lattice of size $M \times L$, then adding one column,



and then adding columns successively until the full width of L is reached. If Z_M is the partition function for a lattice of M links with fixed spins at the boundaries, the formula for adding one additional column of links is

$$Z_{M+1}(\{s_{0j}\}, \{s_{M+1j}\}) = \sum_{s_{Mj}} Z_M(\{s_{0j}\}\{s_{Mj}\}) \times e^{\beta J \sum_j (s_{Mj} s_{M+1j} + s_{M+1j} s_{M+1j+1})}$$

We can write this formula as

$$Z_{M+1} = Z_M \cdot T$$

where the transfer matrix T is a matrix on the space of spin configurations in a column. Each column has L spins, so there are 2^L configurations and thus T is a $2^L \times 2^L$ matrix. It is not so easy to diagonalize such a matrix, but it can sometimes be done.

The matrix T I have defined here is not self-adjoint, but we can recast the problem to construct a self-adjoint transfer matrix, as we did in the case of the 1 dimensional Ising model in a magnetic field. Here, the partition function with periodic boundary conditions is given by

$$Z_{N,\mathrm{pe}} = \mathrm{tr}\, T^N$$

with T the modified transfer matrix

$$T(\{s_{Mj}\}, \{s_{M+1\,j}\}) = e^{\frac{\beta J}{2}\sum_j s_{Mj}s_{Mj+1}}\, e^{\beta J\sum_j s_{Mj}s_{M+1\,j}}\, e^{\frac{\beta J}{2}\sum_j s_{M+1\,j}s_{M+1\,j+1}}$$

The transfer matrix T is a sort of evolution kernal that propagates the spin configuration forward by one column. Thus, it is reasonable to recast T as

$$T = e^{-\beta\mathcal{H}}$$

where $\mathcal{H}$ is a Hamiltonian on a 1 dimensional spin system with variables s_{Mj}. More generally, for a d dimensional problem spin problem, $\mathcal{H}$ would be the Hamiltonian of a $(d-1)$ dimensional quantum spin system.

Using this intuition, we can try to guess the spectrum of T by assuming that $\mathcal{H}$ has a spectrum that looks like that of a multiparticle quantum system. The Ising model has a symmetry P that commutes with T and $\mathcal{H}$. It is helpful to classify states by their parity quantum number.

Let us first make the simplest assumption, that $\mathcal{H}$ has a unique ground state $|v_0\rangle$ with $P = +$. This state gives the largest eigenvalue of T. Let S_j be the operator that flips the spin on the site Mj. This operator is flipped by P,

$$\mathcal{P} S_j = -S_j \mathcal{P}$$

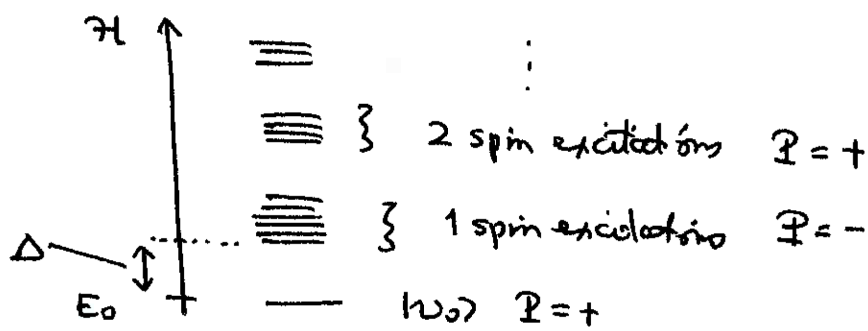
Thus,

$$S_j |v_0\rangle$$

is a state of odd parity. There should be many independent states of this type, corresponding to single-spin excitations at different points along the surface. These would probably be the lowest-energy excitations of $|v_0\rangle$. At somewhat higher energy, we would find states with two flipped spins and even parity, for example,

$$S_i S_j |v_0\rangle$$

The complete spectrum of $\mathcal{H}$ would then take the form



The largest eigenvalue of T would be

$$\lambda_0 = e^{-\beta E_0}$$

If Δ is the gap between the ground state and the lowest state with $P = -$, then the largest odd-parity eigenvalue of T would be

$$\lambda_1 = e^{-\beta(E_0+\Delta)}$$

With this information, we can repeat the argument given above for the form of the spin-spin correlation function. The argument goes through, under the assumptions above, in any dimension. We find

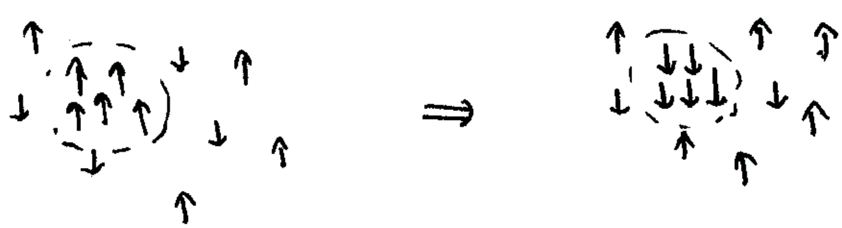
$$\langle S_n S_m \rangle \sim (\text{const}) \cdot \left(\frac{\lambda_1}{\lambda_0}\right)^{|m-n|}$$

$$\sim (\text{const}) \cdot e^{-|m-n|\beta\Delta}$$

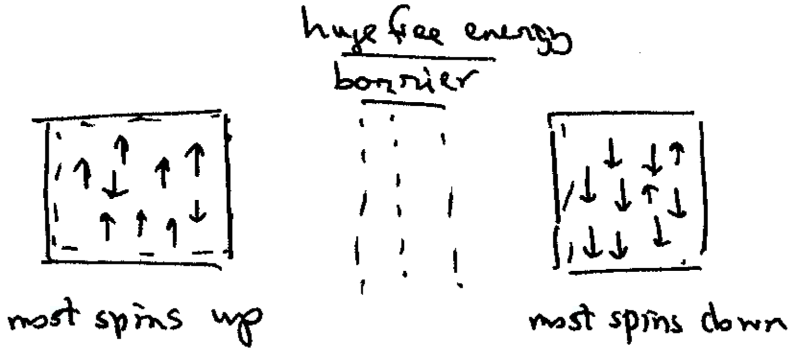
That is, if $\Delta > 0$, there is an exponential decay of correlations. Even if we pin the spin orientation to be up at some position on the lattice, the average spin orientation far away falls to zero. Then there is no net magnetization.

This picture would seem to completely exclude the presence of spontaneous magnetization. But, if we think a little harder, we will see that magnetization can occur we make a slightly different assumption about the lowest energy states of $\mathcal{H}$.

The microcanonical and canonical ensembles sum *symmetrically* over all spin configurations. However, it could happen that the sum divides into two regions that are well separated from one another by regions of very low probability. Cooperative interactions drive neighboring spins to point parallel to one another. At high temperature, spins are quite randomized. A small cluster of spins might all point up, but with some expense of free energy, it is possible to turn these spins over.



However, at sufficiently low temperature, the correlated clusters of spins might become very large, and the expense of free energy needed to turn the cluster over might be prohibitive. Then the phase space of spin configurations will break up into two parts:



To an excellent approximation, we can diagonalize $\mathcal{H}$ over the Hilbert space of states with dominantly spin up and, separately, diagonalize $\mathcal{H}$ over the space of states with dominantly spin down. Let $|\Uparrow\rangle$ and $|\Downarrow\rangle$ be the ground states found in these diagonalizations. By parity, $|\Uparrow\rangle$ and $|\Downarrow\rangle$ will have the same energy. For a very large system, $|\Uparrow\rangle$ and $|\Downarrow\rangle$ will be orthogonal,

$$\langle \Uparrow | \Downarrow \rangle \sim \exp[-(\text{const})\cdot N]$$

In fact, in the limit as $N \to \infty$, we will find that, for any local operator $\mathcal{O}$,

$$\langle \Uparrow | \mathcal{O} | \Downarrow \rangle \to 0 \quad \text{as} \quad N \to \infty$$

Such a relation is called a *superselection rule.*

In this situation, $\mathcal{H}$ has two degenerate ground states

$$\mathcal{H}|\Uparrow\rangle = E_0 |\Uparrow\rangle \qquad \mathcal{H}|\Downarrow\rangle = E_0|\Downarrow\rangle$$

Parity acts by

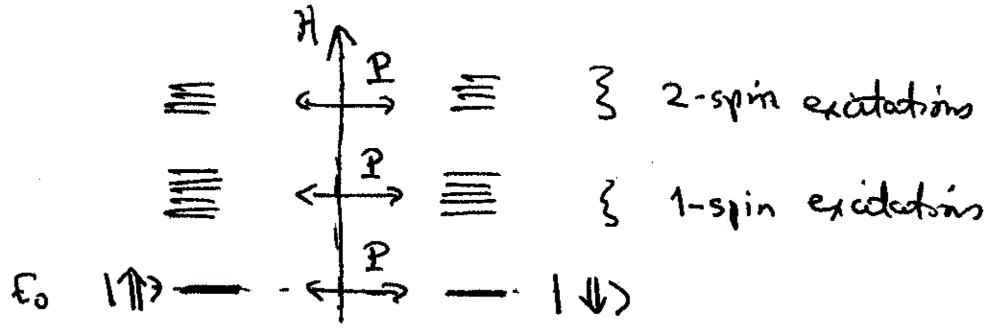
$$\mathcal{P}|\Uparrow\rangle = |\Downarrow\rangle \qquad \mathcal{P}|\Downarrow\rangle = |\Uparrow\rangle$$

It is still true that we can formally construct states of even and odd parity by writing

$$\frac{1}{\sqrt{2}}\left(|\Uparrow\rangle \pm |\Downarrow\rangle\right)$$

However, because of the superselection rule, this is irrelevant. We will obtain a clearer physical picture by working with $|\Uparrow\rangle$ and $|\Downarrow\rangle$ separately. This corresponds to summing, not over the whole set of states in the canonical ensemble, but only over those states that can be reached from $|\Uparrow\rangle$ or $|\Downarrow\rangle$ by the application of local operators.

In this picture, the spectrum of $\mathcal{H}$ has the form



The states $|\Uparrow\rangle$ and $|\Downarrow\rangle$ do not have definite parity, so

$$S_j |\Uparrow\rangle$$

has overlap both with $|\Uparrow\rangle$ and with single-spin excitations built on $|\Uparrow\rangle$.

Now we can redo the analysis of the spin-spin correlation function, and we will get a somewhat different result. If we compute $\langle s_{ni} s_{mi}\rangle$ by averaging in the partial ensemble of states with mainly spin up, we have

$$\langle s_{ni}\, s_{mi}\rangle = \frac{\langle\Uparrow|\, S_i\, T^{m-n}\, S_i\, |\Uparrow\rangle}{\lambda_0^{m-n}}$$

This evaluates to

$$\langle s_{ni} s_{mi} \rangle = \frac{1}{\lambda_0^{m-n}} \Big\{ \langle \Uparrow | s_i | \Uparrow \rangle \lambda_0^{m-n} \langle \Uparrow | s_i | \Uparrow \rangle + \sum_{\text{excited states } a} \langle \Uparrow | s_i | ex(a) \rangle \lambda_a^{m-n} \langle ex(a) | s_i | \Uparrow \rangle$$

$$\sim |\langle \Uparrow | s_i | \Uparrow \rangle|^2 + |\langle \Uparrow | s_i | ex(1) \rangle|^2 \left(\frac{\lambda_1}{\lambda_0} \right)^{m-n} + \cdots$$

so that, finally,

$$\langle s_{ni} s_{mi} \rangle \cong \langle s \rangle^2 + (\text{const}) \cdot e^{-|m-n|/\xi(T)}$$

In this model, correlations exponentially decay down to a *nonzero* constant. This constant is equal to the square of

$$\langle s \rangle = \langle \Uparrow | s_i | \Uparrow \rangle$$

that is, the thermal average magnetization per site.

Repeating this analysis in the partial ensemble of states with mainly spin down, we would find the same result. The constant term is now

$$\left(\langle \Downarrow | s_i | \Downarrow \rangle \right)^2$$

which equals $(\langle \Uparrow | S_j | \Uparrow \rangle)^2$ by parity symmetry. The full canonical average of the spin-spin expectation value is the average of these two identical expressions, and so has the same form.

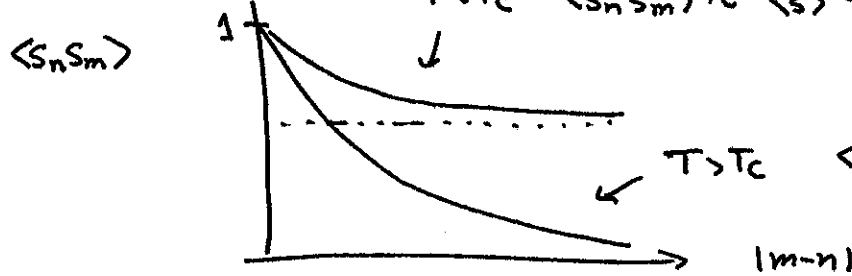
The result

$$\langle s_n s_m \rangle \longrightarrow \langle s \rangle^2 \neq 0 \quad \text{as} \quad |n-m| \to \infty$$

is called *Long Range Order*. This signals the appearance of a spontaneous magnetization at zero field. If we can compute $\langle s_n s_m \rangle$ in a completely symmetrical analysis, for example, an exact sum over the canonical ensemble, we could find this result and learn that the system has degenerate ground states and nonzero magnetization.

In mean field theory, we found that, at $h = 0$, the system is magnetized at low temperatures, below the critical point, and is unmagnetized at high temperatures. The spin-spin correlation function then would have the following behaviors as a function of temperature:

$$T < T_c \qquad \langle s_n s_m \rangle \sim \langle s \rangle^2 + C\, e^{-|m-n|/\xi(T)}$$



$$T > T_c \qquad \langle s_n s_m \rangle \sim C\, e^{-|m-n|/\xi(T)}$$

It makes sense that $\langle s_n s_m \rangle$ increases monotonically as the temperature decreases. The evolution in this figure has that property.

At this point, you might still doubt that any system actually does have long-range order or spontaneous magnetization at a temperature $T > 0$. We have shown that this does not happen in the 1 dimensional Ising model. However, in 1936, *Peierls* gave a proof that the 2 dimensional Ising model has spontaneous magnetization at a sufficiently low temperature. I will now present Peierls' argument, in the form given by Griffiths (Phys. Rev. 136, A437 (1964)).

To make the argument, consider a 2 dimensional Ising model of size $L \times L$, $L^2 = N$, with all spins *up* on the boundary. Spins near the boundary will be preferentially oriented up. However, if there is no long-range order, $\langle s_{ij} \rangle$ will go exponentially to zero away from the boundary. Then we will find

$$M = \sum_{ij} \langle s_{ij} \rangle \sim L^1$$

or

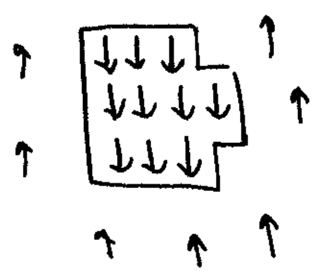
$$M/N \sim \frac{1}{L} \to 0 \quad \text{as} \quad N \to \infty$$

On the other hand, if $\langle s_{ij} \rangle$ tends to a constant in the interior, then

$$M/N \longrightarrow (\text{const}) \qquad \text{as} \quad N \to \infty$$

and we have a nonzero spontaneous magnetization.

With the boundary conditions we have chosen, all spins in the lattice will be up at zero temperature. At low but finite temperature, neighboring spins will be correlated. We can think of correlated down spins as occuring in clusters. For example, a cluster of down spins in a larger group of up spins looks like



Each cluster is surrounded by mismatched bonds. Each mismatched bond has a statistical weight

$$e^{-2\beta J}$$

relative to that of a matched bond. The cluster shown above is surrounded by 14 mismatched bonds, so it has a weight

$$e^{-14\cdot 2\beta J}$$

However, there are many arrangements of clusters of down spins, so we must count the entropy associated with these configurations. If there is sufficient entropy, the system will fill with clusters of down spins and will be macroscopically disordered.

To assess this possibility, we need to sum over, or at least bound, the configurations of down spins and mismatched bonds.

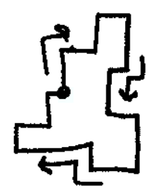
A cluster bounded by b mismatched bonds encloses at most

$$\left(\frac{b}{4}\right)^2$$

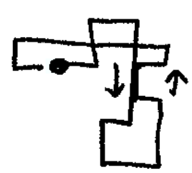
down spins. From this observations, we can bound the number of down spins. Let $m(b)$ be the number of clusters with b mismatched bonds on its boundary. Then

$$N_\downarrow \leq \sum_{b=4,6,8} \frac{b^2}{16} e^{-b\cdot 2\beta J} m(b)$$

Here is a process that generates a loop of mismatched bonds: Start at any site and add a mismatched bond on any one of the 4 directions. From that site, add a mismatched bond in any of 3 directions. Continue in this way, making a choice of one of 3 directions at every site, until we return to the original point. The last choice is determined, and we should divide by b because we generate the same path starting from any point on the loop.



Counting loops in this way *overestimates* $m(b)$, because it does not take into account that we might cross an earlier path or backtrack along the same link



Thus,

$$m(b) \leq N \cdot 4 \cdot 3^{b-2}/b$$

Combining these estimates,

$$N_{\downarrow} \leq \frac{N}{36} \sum_{b=4,6,8} b\, 3^{b}\, e^{-2\beta J b}$$

This is a convergent series, and it can be summed,

$$N_{\downarrow} \leq \frac{N}{18} \kappa^{4} \frac{2-\kappa^{2}}{(1-\kappa^{2})^{2}}$$

where

$$\kappa = 3\, e^{-2\beta J}$$

The parameter κ becomes very small as $T \to 0$. We can find a $\overline{T}$ such that

$$\frac{1}{\sqrt{2}} = \overline{\kappa} = 3\, e^{-2\overline{\beta} J}$$

and so

$$N_{\downarrow} \leq \frac{5}{36} N$$

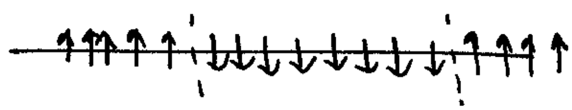
Then, at this temperature and at all lower temperatures,

$$M = N_{\uparrow} - N_{\downarrow} = N - 2N_{\downarrow} > \frac{13}{18} N > 0 \ !$$

This proves that the 2 dimensional Ising model has long-range order over a range of temperatures

$$T < \bar{T}$$

Notice that this argument does not go through in 1 dimension. In 1 dimension, a cluster of down spins requires only two mismatched bonds,



This means that *one flipped spin* is enough to destroy long-range order. Even if the probability of finding a down spin is small, we will eventually find one along a large chain. Thus, we cannot expect long-range order in 1 dimension. However, the Ising model has long-range order at sufficiently low temperature in any dimensionality $d > 1$.

It is quite remarkable that this sum over borders of down spin clusters is a convergent sum for sufficiently low temperature. That means that we can use this picture to obtain a series expansion for the free energy of the Ising model that actually gives us quantitative information at low temperatures. It is straightforward to obtain this series expansion. We simply account the possible configurations of down spins, weighting each configuration by

$$e^{-2\beta J \cdot b}$$

Qualitative conclusions from this series expansion—for example, the statements that there is a nonzero spontaneous magnetization and that there is an exponential decay of spin correlations to this nonzero value—are justified rigorously.

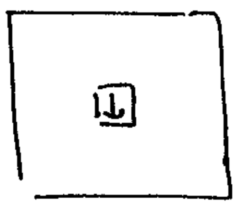
The configuration with *no* down spins contributes

$$Z = e^{+2N\beta J}$$

Extracting this factor, we have

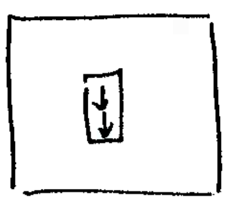
$$b = 0 \qquad 1$$

There are N configurations with 1 down spin



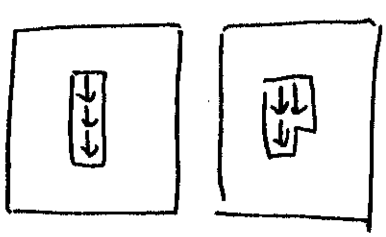
$$b = 4 \qquad N e^{-4 \cdot 2\beta J}$$

There are $2N$ configurations with 2 adjacent down spins, counting the two different orientations.



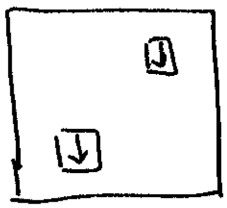
$$b = 6 \qquad 2N e^{-6 \cdot 2\beta J}$$

There are $6N$ configurations with 3 adjacent down spins and 8 mismatched bonds



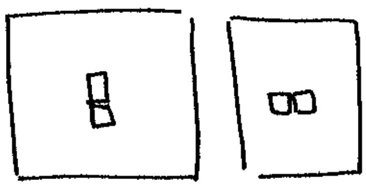
$$b = 8 \qquad (2+4) N e^{-8 \cdot 2\beta J}$$

In the same order in $e^{-2\beta J}$, we have configurations with 8 mismatched bonds in which there are two down spins that are not adjacent,



$$b = 8 \qquad \frac{N}{2}(N-1)\, e^{-8\cdot 2\beta J}$$

Naively, there are $N(N-1)2$ such configurations. However, we must also subtract $2N$ configurations in which the spins are at adjacent locations.



$$-2N\, e^{-8\cdot 2\beta J}$$

Adding all of these contributions, we find

$$Z = e^{2N\beta J}\Big(1 + N e^{-8\beta J} + 2N e^{-12\beta J} + \left(\tfrac{1}{2}(N^2-N) + (6-2)N\right) e^{-16\beta J} + \dots$$

The term in parentheses is the *exponential* of an *extensive* quantity. It can be shown that this is true to all orders. This is a less obvious example of the principle of *exponentiation of the disconnected diagrams.* The final result is

$$Z = \exp\left[2N\beta J + N \quad e^{-8\beta J} + 2e^{-12\beta J} + \frac{7}{2} e^{-16\beta J} + \dots\right]$$

so that

$$F = -2NJ - NT\left(e^{-8\beta J} + 2e^{-12\beta J} + \frac{7}{2} e^{-16\beta J} + \dots\right)$$

With the help of a computer, it is possible to carry out this graphical enumeration to about 20 terms. This gives a precise computation of F at low temperatures.

There is a similar method that allows us to compute F at high temperatures. The partition function of the Ising model is a product of factors

$$e^{\beta J s_i s_j}$$

Since $s_i s_j$ equals either +1 or -1, this can be written

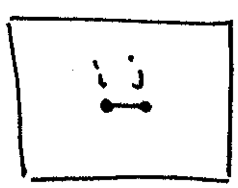
$$\begin{aligned} &\cosh \beta J + s_i s_j \sinh \beta J \\ &= \cosh \beta J \, (1 + s_i s_j \tanh \beta J) \end{aligned}$$

Then

$$Z = (\cosh \beta J)^{2N} \sum_{\{s_i = \pm 1\}} \prod_{\langle ij \rangle} (1 + s_i s_j \tanh \beta J)$$

The indicated product runs over bonds on the lattice.

At high temperature, $\tanh \beta J$ is small. To expand in this parameter, we use the term 1 on most bonds and the second term on just a few bonds. I will indicate these bonds by coloring them. A configuration with m colored bonds is of order $(\tanh \beta J)^m$. However, not all configurations of colored bonds contribute. Notice that



$$= \sum_{s_i, s_j} s_i s_j \tanh \beta J$$

This contribution is *zero*, because

$$\sum_{s_i} s_i = 1 + (-1) = 0$$

In general, the only contributions that do not vanish are those that give

$$1, \quad s_i^2 \quad \text{or} \quad s_i^4$$

on each lattice site. That is, only those colorings are nonzero that have an *even* number of colored bonds attached to each lattice site.

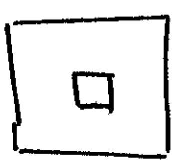
The description of the Ising model by this series implies that there is no spontaneous magnetization and that spin correlations fall exponentially to zero.

With this understanding, we can write the perturbation series for Z in powers of $\tanh \beta J$. The expression has the structure

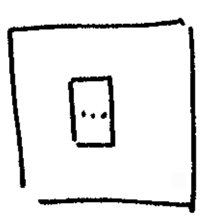
$$Z = 2^N (\cosh \beta J)^{2N} \left[1 + \cdots \right]$$

where the quantity in parentheses is the series

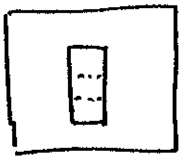
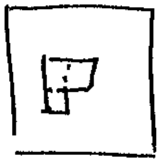
$$1$$



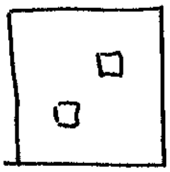
$$N (\tanh \beta J)^4$$



$$2N (\tanh \beta J)^6$$

$$6N\,(\tanh\beta J)^8$$



$$\left[\tfrac{1}{2}N(N-1) - 2N\right]\,(\tanh\beta J)^8$$

Thus

$$Z = 2^N(\cosh\beta J)^{2N}\Big(1 + N(\tanh\beta J)^4 + 2N(\tanh\beta J)^6$$
$$+ \left(\frac{N(N-1)}{2} + (6-2)N\right)(\tanh\beta J)^8 + \cdots\Big)$$

This is exactly the same series that we found in the previous analysis, and it exponentiates in the same way.

$$Z = \exp\Big[\,N\log 2\cosh^2\beta J$$
$$+ N\left((\tanh\beta J)^4 + 2(\tanh\beta J)^6 + \tfrac{7}{2}(\tanh\beta J)^8 + \cdots\right)\Big]$$

so that

$$F = -NT\log 2\cosh^2\beta J$$
$$- NT\left[(\tanh\beta J)^4 + 2(\tanh\beta J)^6 + \tfrac{7}{2}(\tanh\beta J)^8 + \cdots\right]$$

It is not difficult to see that, for the 2 dimensional Ising model, the two series are completely equivalent. Every configuration of down spins in the first example is surrounded by mismatched bonds in such a way that each vertex of bonds has an even number of bonds attached to it. Conversely, in every arrangement of bonds with an even number of bonds attached to each vertex, the bonds for closed loops, and we can imagine that these loops are filled with down spins. We can then relate the expressions for the free energy that come from these analyses. From the low temperature series, we found

$$F = -2NJ - NT\, f(e^{-2\beta J})$$

From the high temperature series, we found

$$F = -NT \log(2\cosh^2\beta J) - NT\, f(\tanh\beta J)$$

The function $f(x)$ that appears in the two cases is *the same convergent series expansion* and therefore is *the same function* in the two formulae. *Kramers and Wannier* discovered this and drew a two interesting conclusions.

First, the relationship between the two expansions is a *duality* that maps high temperature spin systems to low temperature spin systems and vice versa. This *Kramers-Wannier duality* relates spin systems at β_1 and β_2 such that

$$e^{-2\beta_1 J} = \tanh\beta_2 J$$

Note that $\beta_1 \to 0$ implies $\beta_2 \to \infty$ and vice versa.

Second, we know that the 2 dimensional Ising model has a critical point at which $F(\beta)$ has singularities, and thus these singularities must appear in both expressions for F. The simplest assumption is that $F(\beta)$ is singular at one temperature β_c or $T_c = 1/\beta_c$. This must correspond to a singularity in the function $f(x)$ at a value x_c such that both

$$x_c = e^{-2\beta_c J} \quad \underline{\text{and}} \quad x_c = \tanh\beta_c J$$

This is an equation that we can solve for β_c.

$$e^{-2\beta_c J} = \frac{1 - e^{-2\beta_c J}}{1 + e^{-2\beta_c J}}$$

$$\text{or} \qquad x_c^2 + 2x_c - 1 = 0$$

There is one positive solution,

$$x_c = \sqrt{2} - 1 = \frac{1}{\sqrt{2} + 1}$$

Then

$$2\beta_c J = \log(\sqrt{2} + 1)$$

From this logic, we have located the critical point,

$$T_c = \frac{2J}{\log(\sqrt{2}+1)} = 2.27 \ J$$

This value can be compared to the mean field theory prediction

$$T_c \big|_{mft} = 4J$$

In 1944, in a tour de force calcuation, *Onsager* exactly diagonalized the transfer matrix for the 2 dimensional Ising model. This exact solution is beautifully described in Schultz, Mattis, and Lieb, Rev. Mod. Phys. 36, 856 (1964). The exact solution has the following properties:

- There is exactly one singularity of $F(T)$, and it occurs at the temperature T_c corresponding to the Kramers-Wannier self-dual point $x_c = \sqrt{2} - 1$.

- The specific heat C_V has a singularity at T_c. Mean field theory predicts a discontinuity in C_V at T_c; Onsager found instead

$$C_V \sim \log |T - T_c|$$

- At $T = T_c$, the spin-spin correlation function decays as a power law

$$\langle s_m s_n \rangle \sim \frac{1}{|m-n|^{\frac{1}{4}}}$$

- At $T < T_c$ and zero field, there is a nonzero spontaneous magnetization. This was computed by C. N. Yang in another very nontrivial computation. Yang found that

$$M \sim (T_c - T)^{1/8} \qquad T \lesssim T_c$$

The exact solution of the 2 dimensional Ising model confirms the idea from mean field theory that systems with cooperative behavior have critical points with singularities in thermodynamic functions. However, the exact solution differs quantitatively from the results of mean field theory, not only in such quantities as the value of T_c but also in the values of the *exponents* involved in the singular factors. Clearly, there is much more here that we need to understand.