May 27

Critical Exponents (continued)

In the previous lecture, we began our analysis of scale-invariant correlations in the vicinity of a critical point. I introduced the goal of computing the *critical exponents* that give the non-analytic behavior of thermodynamic functions in this region. As a tool for this computation, I introduced the *Renormalization Group* (RG) transformation. This is a transformation that changes the underlying length scale of a model by summing or intergrating over short-distance degrees of freedom.

Using the RG transformation, we analyzed correlations in the 1- and 2-dimensional Ising model. For the 1-dimensional Ising model, we found an explicit solution in the form of a recursion formula

$$(\beta J)' = f(\beta J)$$

This relation always transformed a value of (βJ) to a *smaller* value of (βJ). Iterating the RG transformation, we always found a model at large β or high temperature. Such a model has short-range correlations and $\langle S_i \rangle = 0$. So it must be true that, in the original model, we are in the disordered phase at any finite temperature.

For the 2-dimensional Ising model, the RG transformation was not so simple. A direct implementation of the RG transformation produced a more complicated model with additional 4-, 6-, and higher-spin interactions. We analyzed this model by making approximations to the RG transformation that reduced it to a one-parameter recursion similar to that in 1 dimension. In this case, we found that, for sufficiently small values of (βJ), that parameter was transformed to smaller values, indicating the disordered phase. However, for sufficiently large values of (βJ), that parameter was transformed to larger values, indicating a phase with ordering and $\langle S_i \rangle \neq 0$. The boundary between these two regions was a *fixed point* of the RG transformation. We saw that this point could be associated with the critical point, and we used the properties of the RG transformation near this point to compute the critical exponent ν.

In this lecture, I will continue this analysis by discussing the *full* RG transformation, including all possible operators that might be induced in the calculation. This will give a complete and very beautiful picture of scale transformations in statistical mechanics, due to *Kenneth Wilson.*

A full analysis of the RG transformation for a statistical mechanics model will bring in all possible local interactions. I propose that we have no fear and simply proceed directly to build a formalism on the space of all possible Hamiltonians.

Consider, then, statistical mechanics models on a lattice of lattice spacing a, or another system with a minimal length scale a. In a fluid model, for example, a might be the size of the hard-core repulsive interaction. In this discussion, I will use the language of magnets for definiteness. Write the partition function of the model schematically as

$$Z = \sum_{s} e^{-\mathcal{H}[s]}$$

In the discussion to follow, β will be included in $\mathcal{H}[S]$. The RG transformation will be a general motion in this space

$$\mathcal{H}[s] \longrightarrow \mathcal{H}'[s]$$

Let $\mathcal{H}_0$ be a particular choice of the Hamiltonian for this system. Then we can parametrize a general Hamiltonian near $\mathcal{H}_0$ as

$$\mathcal{H}[s] = \mathcal{H}_0[s] + \sum_{m} c_m \underbrace{\sum_{i}}_{\text{pts on lattice}} \mathcal{O}_m(i)$$

For example, $\mathcal{H}_0$ might be the Ising model Hamiltonian and the list $\mathcal{O}_m$ might include local 4- and 6-spin interactions.

We can now define an RG transformation on this space of Hamiltonians, for example, by summing over every other spin and writing the result as the exponential of a Hamiltonian,

$$\sum_{s_x} e^{-H[s, s_x]} = e^{-\mathcal{H}'[s]}$$

If $\mathcal{H}'[S]$ remains in the neighborhood of $\mathcal{H}_0$, we can write this transformation as a transformation of the coefficients C_j

$$\mathcal{H}'[s] = H_0[s] + \sum_m c'_m \sum_i \mathcal{O}_m(i)$$

so that we have a mapping

$$c_m \longrightarrow c'_m[c_n]$$

In the previous lecture, we saw examples of Hamiltonians that were *fixed points* of the RG transformation,

$$\mathcal{H}[s] = \mathcal{H}'[s] = \mathcal{H}_*[s]$$

Systems described by such Hamiltonians have *the same* spin correlations at all distance scales. That is, these systems are exactly scale-invariant. I argued in the previous lecture that such a fixed-point Hamiltonian would represent a critical point.

It makes sense to choose the fixed-point Hamiltonian $\mathcal{H}_*$ as the origin of coordinates above. A general Hamiltonian is then represented as an expansion about the fixed point.

$$\mathcal{H}[s] = \mathcal{H}_*[s] + \sum_m c_m \sum_i \mathcal{O}_m(i)$$

At the fixed point $C_m = 0$ maps to $C'_m = 0$. Then the relation between C_m and C'_m is approximately *linear* for small values of the C_m,

$$C'_m = X_{mn} C_n + \mathcal{O}(C^2)$$

For very small deviations from the critical point, the step size in the RG transformation becomes very small and we can approximate the RG motion by a continuous evolution. If λ is the change of scale in the RG transformation, we then write

$$\lambda \frac{d}{d\lambda} C_m(\lambda) = M_{mn} C_n(\lambda) + \mathcal{O}(C^2)$$

This is a *continuous flow* in the space of possible Hamiltonians.

To analyze this system, we diagonalize the matrix $\mathcal{M}_{mn}$ and identify the characteristic directions in the flow about the critical point. Let the eigenvectors v_{jn} and eigenvalues b_j be given by

$$M_{mn} v_{jn} = b_j v_{jn}$$

If we identify

$$\mathcal{O}_j(i) = v_{jm} \mathcal{O}_m(i)$$

as the eigenoperators, we can rewrite the expansion of a general $\mathcal{H}[S]$ as

$$\mathcal{H}[S] = H_* + \sum_j \alpha_j \sum_i \mathcal{O}_j(i)$$

The evolution of this Hamiltonian under the RG transformation is

$$\mathcal{H}[s] \to H_* + \sum_j \alpha_j \lambda^{b_j} \sum_i \mathcal{O}_j(i)$$

A term with $b_j > 0$ increases in importance under the RG transformation; a term with $b_j < 0$ decreases.

We can classify the eigenoperators by the sign of the associated b_i. The following terms are applied:

- If $b_j > 0$, $\mathcal{O}_j$ is a *relevant operator*. A perturbation by $\mathcal{O}_j$ *grows* in importance at large scales.
- If $b_j < 0$, $\mathcal{O}_j$ is an *irrelevant operator*. A perturbation by $\mathcal{O}_j$ *decreases* in importance at large scales.
- If $b_j = 0$, $\mathcal{O}_j$ is a *marginal operator*. To decide whether a perturbation by $\mathcal{O}_j$ is important at large scale, we need to examine the nonlinear terms in the RG evolution equation.

Many interesting physical systems contain marginal operators. A typical form of the evolution equation for a marginal perturbation is

$$\lambda \frac{d}{d\lambda} C(\lambda) = -a C^2(\lambda)$$

The solution to this equation is

$$C(\lambda) = \frac{C(1)}{1 + a C(1) \log \lambda}$$

If $a > 0$, C becomes irrelevant at large distances, but very slowly as a function of scale. If $a < 0$, a value of C that is small at small scales can grow in importance and eventually dominate the dynamics. This second case is called *asymptotic freedom.* It is seen, for example, in the *Kondo problem*, an isolated magnetic impurity coupled antiferromagnetically to an electron Fermi gas. It is also possible to have an *exactly marginal* direction

$$\lambda \frac{d}{d\lambda} c(\lambda) = 0$$

Such a direction appears in 2-dimensional models, for which there are examples in which Hamiltonians $\mathcal{H}(g)$ are equivalent to systems of free fermions or bosons for any value of g. However, in the rest of this lecture, I will ignore marginal perturbations and consider only the cases of relevant and irrelevant perturbations.

The largest values of the b_i are related to critical exponents. To discuss the connection, I will recall that critical points typically involve spontaneous breaking of a symmetry G. The original Hamiltonian $\mathcal{H}_0$ respects the symmetry; that is

$$[G, \mathcal{H}_0] = 0$$

The RG transformation can be arranged also to respect the symmetry. Then a perturbation $\mathcal{O}_m$ can transformation only into operators that transform in the same way under G. If we write the original Hamiltonian $\mathcal{H}_0$ as a perturbation of $\mathcal{H}_*$, the perturbations can only involve operators that commute with G, and the RG evolution will only generate operators that commute with G.

Among these operators, there is an eigenoperator with the largest positive value of b_j. Call this operator $\mathcal{O}_t$ and the corresponding eigenvalue b_t. Then, an original Hamiltonian

$$\mathcal{H}_0 = \mathcal{H}_* + c_t \sum_i \mathcal{O}_t(i)$$

evolves, after a rescaling by λ, to

$$\mathcal{H}_* + c_t \lambda^{b_t} \sum_i \mathcal{O}_t(i)$$

If the original Hamiltonian is close to the critical point, the coefficient C_t will be small, proportional to

$$C_t = a \cdot t \qquad t = \frac{T - T_c}{T_c}$$

After sufficient RG evolution, the coefficient $C_t(\lambda)$ will become large. When $C_t(\lambda)$ is of order 1, we have reached a model far from the critical point. This model will have short-range correlations. That is, the correlation length ξ will be of order 1 in *current* lattice units

$$\xi \sim \lambda a \qquad \text{for} \qquad t\lambda^{b_t} \sim 1$$

Thus

$$\xi \sim t^{-1/b_t}$$

This implies

$$\nu = 1/b_t$$

This relation will apply, further, to any Hamiltonian in the vicinity of $\mathcal{H}_*$ that commutes with G. Perturbations by irrelevant operators will become unimportant at large distances, so they will not affect the conclusion. Some perturbations mAY grow with RG evolution, but the strongest growth will be in the direction of $\mathcal{O}_t$.

We can also analyze the effect of adding to $\mathcal{H}_0$ perturbations that do not commute with G. For example, in a magnetic system, we can add a small external field

$$\Delta H = -h \sum_i s_i$$

Here h multiplies an operator that transforms nontrivially under G. Under the RG evolution, this operator will general other operators with the same transformation law. Among these, there will be an eigenoperator with the largest positive value of b_j. Call this operator $\mathcal{O}_h$ and the corresponding eigenvalue b_h. After a large amount of RG evolution, the perturbation above will evolve to

$$\Delta H = -h \lambda^{b_h} \sum_i \mathcal{O}_h(i)$$

The spin-spin correlation function in the original model obeys the relation

$$\frac{\partial^2}{\partial h^2} \log Z = (\text{Volume}) \cdot \int d^3x \quad \langle S(x) S(0) \rangle$$

The same equation holds for the model obtained from RG evolution. By comparing the two equations, we can see the rescaling of the correlation function in the region of scale-invariant behavior. We must take into account the rescaling of the volumes. Then

$$\langle S(x) S(0) \rangle = (\lambda^{-d} \lambda^{b_h})^2 \langle S(x/\lambda) S(0) \rangle$$

Thus, we identify

$$A = d - b_h$$

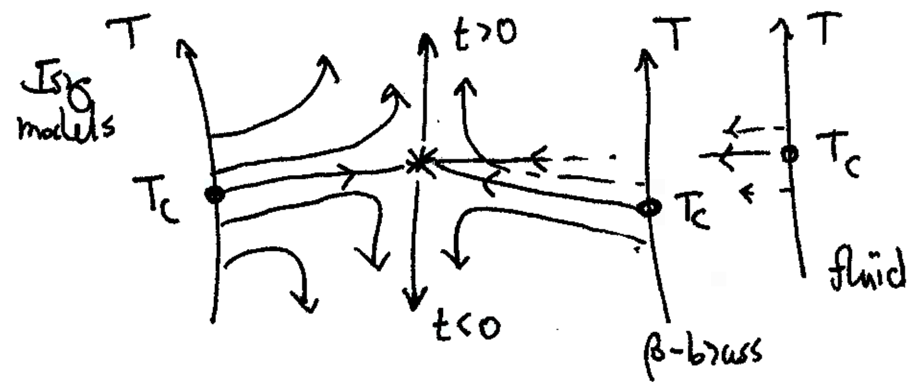
where A is the spin-rescaling parameter defined at the beginning of the previous lecture. This is related to the critical exponent η, as explained there.

I have now given a prescription for computing the critical exponents η and ν in terms of the eigenvalues of the RG transformation in the vicinity of a fixed point,

$$A = \frac{d-2+\eta}{2} = d - b_h \qquad B = \frac{1}{\nu} = b_t$$

We saw in the previous lecture that all of the other critical exponents can be computed from η and ν. So, we have now reduced the problem of computing critical exponents to that of analyzing the stability behavoir of the RG transformation.

If a fixed point has exactly one relevant perturbation that commutes with G, we have *universality*, which I defined previously as the equality of critical exponents for very different statistical mechanics models. For example, Ising magnets and lattice Ising models, order-disorder transitions in alloys such as β-brass, and the critical end-point in fluids have the same critical exponents. The universality of these exponents would be explained by the following picture of RG evolution:



Each model gives a line in the space of Hamiltonians, parametrized by the temperature. On each line, there is a critical temperature. Under the RG transformation, these critical Hamiltonians flow to a common fixed point $\mathcal{H}_*$. Hamiltonians very close the critical ones flow to the very near vicinity of $\mathcal{H}_*$. The fixed point is unstable in one direction, corresponding to a perturbation by it leading operator $\mathcal{O}_t$. All three systems then share the same value of ν. The perturbations of each Hamiltonian by a symmetry-breaking field evolve to the same operator $\mathcal{O}_h$, so these systems also share the same value of η.

Wilson took these ideas one step further. I have explained above that, quite generally, systems with the same Landau theory share the same critical exponents. So, it is interesting to look for the fixed point $\mathcal{H}_*$ in the space of Landau theories. I will now discuss an implementation of this idea by defining an RG transformation on Landau theories. For simplicity, I will consider Landau theories with Ising symmetry, with a one-component field $M(x)$ such that the Hamiltonian is invariant to $M(x) \to -M(x)$.

A Landau theory is a continuum field theory for which the Hamiltonian is the local Gibbs free energy

$$\mathcal{H}[M] = \int d^d x \left\{ \frac{1}{2}\rho(\vec{\nabla} M)^2 + \frac{1}{2} a t M^2 + \frac{b}{4} M^4 + \cdots \right\}$$

$$t = \frac{(T - T_c)}{T_c}.$$

We can impose a shortest distance on this theory by cutting off divergent integrals at the distance a,

$$\int d^d x \, \frac{1}{x^n} \sim \frac{1}{a^{n-d}} \qquad n > d$$

Equivalently, we can work in Fourier space and consider fields $M(x)$ for which only the Fourier components with

$$|\vec{k}| < \frac{\pi}{a}$$

are nonzero. For precise calculation, I should make these definitions more complete, but definitions of a at this level will suffice for the results that I will derive in this lecture.

In the expression for $\mathcal{H}[M]$, we still have the freedom to rescale the field $M(x)$. It will be useful to fix this freedom by choosing the scale of $M(x)$ to absorb the parameter ρ. Then

$$\mathcal{H}[M] = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla} M)^2 + \frac{at}{2} M^2 + \frac{b}{4} M^4 + \cdots \right\}$$

I will compare this expression to the expression for $\mathcal{H}[M]$ after a step of the RG transformation by insisting that that Hamiltonian also has the coefficient 1 in front of the term $\frac{1}{2}(\nabla M)^2$.

The Hamiltonian

$$H[M] = \int d^d x \; \frac{1}{2}(\vec{\nabla} M)^2$$

is an especially simple one. This model is called the *Gaussian model.* In Fourier space, this Hamiltonian takes the form

$$H[M] = \int \frac{d^d k}{(2\pi)^4} \; \frac{1}{2} \; |M(k)|^2$$

Then, different Fourier modes of $M(x)$ fluctuate independently of one another. Each mode is described by a Gaussian distribution. As we discussed earlier in the course, correlation functions of a Gaussian random variable are given in terms of the 2-point correlation function. For the normalation $\rho = 1$ that I have chosen here, it can be shown that the 2-point correlation function of M is precisely the Green's function of the Laplace equation, with coefficient 1,

$$-\nabla^2 \langle M(x) M(0) \rangle = \delta^{(d)}(\vec{x})$$

The solution of this equation is

$$\langle M(x) M(0) \rangle = \frac{C}{|\vec{x}|^{d-2}}$$

For example,

$$\langle M(x) M(0) \rangle = \begin{cases} \frac{1}{4\pi}\frac{1}{x} & \text{in } d=3 \\ \frac{1}{4\pi^2}\frac{1}{x^2} & \text{in } d=4 \end{cases}$$

It will be interesting to carry out this analysis in a general dimension d.

To implement an RG transformation with a rescaling by λ on Landau theories, I will carry out the following three steps:

1. Integrate over Fourier components of M with

$$\frac{\pi}{\lambda a} < |\vec{k}| < \frac{\pi}{a}$$

2. Rescale distances and momenta by so that the final model is defined over the same region of Fourier space as the original model.
3. Rescale the field $M(x)$

$$x' = x/\lambda \ , \quad k' = \lambda k$$

$$M'(x) = \lambda^{-A} M(x/\lambda)$$

so that the coefficient of the term $\frac{1}{2}(\nabla M)^2$ is equal to 1.

Step 1 is difficult and steps 2 and 3 are easy. However, in the immediate vicinity of the Gaussian model, step 1 is also easy. Since different Fourier modes are not coupled by the Hamiltonian of the Gaussian model, step 1 has no effect when a, b, c, etc. are very small.

Because of this, we can easily find the RG evolution in the neighborhood of the Gaussian model by carrying out steps 2 and 3. This transforms our original form of the Landau theory Hamiltonian to

$$H'[M] = \int d^d x' \, \lambda^d \Big\{ \tfrac{1}{2}(\vec{\nabla}' M')^2 \lambda^{-2} \lambda^{-2A} + \frac{at}{2} \lambda^{-2A} M'^2 + \frac{b}{4} \lambda^{-4A} M'^4 + \cdots \Big\}$$

To set the coefficient of the first term to 1, following the convention given above, set

$$A = \frac{d-2}{2}$$

Then

$$\mathcal{H}'[M] = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla} M)^2 + \frac{at}{2} \lambda^2 M^2 + \frac{b}{4} \lambda^{4-d} M^4 + \cdots \right\}$$

The coefficient of a term with $2p$ powers of M and m derivatives gets a factor

$$\lambda^d \lambda^{-m} \lambda^{p(2-d)} = \lambda^{2p-m-d(p-1)}$$

Then, the various terms in $\mathcal{H}$ are rescaled by

- coefficient of M^2: $(at) \rightarrow (at)\lambda^2$
- coefficient of M^4: $b \rightarrow b\lambda^{4-d}$
- coefficient of M^6: $c \rightarrow c\lambda^{6-2d}$
- coefficient of M^4: $d \rightarrow d\lambda^{8-3d}$
- coefficient of $(\nabla^2 M)^2$: $f \rightarrow f\lambda^{-2}$
- coefficient of M^1: $h \rightarrow h\lambda^{1+d/2}$

In 3 dimensions, there are precisely two relevant operators invariant under G; these are M^2 and M^4. Any operator with derivatives has an eigenvalue that includes a factor λ^{-1} each derivative.

It is interesting that, as d is lowered to $d = 2$, more and more operators of the form M^{2p} become relevant. This means that there should be a more complex classification of fixed points in 2 dimensions. I will discuss this issue further in the next lecture.

We see that the Gaussian model is a fixed point, and that the above operators are eigenoperators. The leading eigenoperator with the symmetry of the model is $\mathcal{O}_t = M^2$, with $b_t = 2$. The leading eigenoperator with the symmetry of the order parameter M is $\mathcal{O}_h = M$, with $b_h = (1 + d/2) = d - (d-2)/2$. This implies that the Gaussian fixed point gives the critical exponents

$$B = 2 \rightarrow \nu = \frac{1}{2} \qquad A = \frac{d-2}{2} \rightarrow \eta = 0$$

These are the critical exponents of Landau theory, which, as we know, are not quite the correct ones.

To find better values of the critical exponents, we should look for another fixed point with nonzero values of the nonlinear couplings in $\mathcal{H}$. Wilson and *Michael Fisher* realized that a new fixed point can be found in the following way: Near $b = 0$, the direction corresponding to the operator M^4 is an unstable direction from the Gaussian fixed point, along which b increases. If we can show that nonlinear terms *decrease* b for larger values of b, there must be an intermediate value of b that is stable under the RG evolution. This is the *Wilson-Fisher fixed point.*

In 3 dimensions, the Wilson-Fisher fixed point occurs at a value of b that is of order 1. However, is we consider a value of d that is closer to $d = 4$, the instability of the M^4 term is weaker. Then it is possible that this instability can be balanced by a small effect proportional to b^2, so that we find a fixed point at a value of b where perturbation theory in b is accurate.

I will now compute the term of order b^2 in the evolution of the coefficient of M^4 near the Gaussian fixed point and show that this picture is correct. I will assume that the dimension d is close to 4,

$$d = 4 - \epsilon$$

In evaluating the b^2 correction, it will be simplest to work explicitly in 4 dimensions.

I consider the Gaussian model perturbed by a small M^4 perturbation. In this case

$$\mathcal{H}[M] = \int d^d x \left\{ \tfrac{1}{2}(\nabla M)^2 + \tfrac{b}{4} M^4 \right\}$$

We can expand the exponential in powers of b,

$$e^{-H[M]} = e^{-\int d^d x \frac{1}{2}(\nabla M)^2} \Big(1 - \tfrac{b}{4} \int d^d x\, M^4$$

$$+ \tfrac{1}{2}\left(\tfrac{b}{4}\right)^2 \int d^d x\, M^4(x) \int d^d y\, M^4(y) + \cdots \Big)$$

I will now evaluate the term of order b^2 in the following way: In each M^4 term, I will consider two of the factors of M to be evaluated in low Fourier components, $k < \pi/\lambda a$, and two of the factors of M to be evaluated in high Fourier components $\pi/\lambda a < k < \pi/a$. Notice that there are no contributions, or, at least, no large ones, in which three of the k values are large and only one is small, since the four k values must sum to zero. There are

$$\frac{4 \cdot 3}{2} = 6$$

ways to choose the factors M with low and high Fourier components in each M^4 factor. Then term of order b^2 becomes

$$+ \frac{1}{2} \frac{b^2}{4 \cdot 4} \cdot 6 \cdot 6 \int d^d x\, d^d y\; M^2(x) \langle M^2(x) M^2(y) \rangle M^2(y)$$

where the expectation value indicates an integral over the high-momentum components of M. This is an expectation value of Gaussian random fields, and so it is given in terms of 2-point correlation functions. There are 2 ways to associate the factors of $M(x)$ and $M(y)$.

$$\langle M^2(x) M^2(y) \rangle = 2\left(\langle M(x) M(y) \rangle\right)^2 = 2\left(\langle M(0) M(y-x) \rangle\right)^2$$

The points x and y are separated by a distance of order λa. After rescaling of distances, these points will be within a and will be associated to the same point. Thus, finally, this expression takes the form

$$+\ \frac{9b^2}{4} \int d^dx\ M^4(x) \quad \int d^dy\ \langle M(y) M(0) \rangle^2$$

Using the value of the 2-point correlation function in 4 dimensions given above, the integral over y becomes

$$S_4 = 2\pi^2$$

$$\int d^4y \left(\frac{1}{4\pi^2} \frac{1}{y^2} \right)^2 = \frac{2\pi^2}{16\pi^4} \int dy\, y^3 \frac{1}{y^4} = \frac{1}{8\pi^2} \int \frac{dy}{y}$$

Then integral over y is over distances from a to λa, so this becomes

$$\frac{1}{8\pi^2} \log \lambda$$

Then the complete expression for the b^2 term is

$$+\ \frac{1}{4} \cdot \frac{9b^2}{8\pi^2} \log \lambda \int d^dx\ M^4$$

The series expansion in b can now be exponentiated

$$1 + \frac{1}{4} \frac{9b^2}{8\pi^2} \log \lambda \int d^dx M^4 = e^{-\int d^dx\ \frac{1}{4} M^2 \left(-\frac{9b^2}{8\pi^2} \log \lambda \right)}$$

The scaling analysis above gave for the evolution of the b term

$$b \to b\lambda^{\epsilon} = 1 + b\cdot\epsilon\cdot\log\lambda$$
$$\epsilon = 4-d$$

Adding that effect and the correction we have just computed, we find for the RG evolution of b near the Gaussian fixed point and near $d = 4$

$$b \to b + \left(b\epsilon - \frac{9b^2}{8\pi^2}\right)\log\lambda + \cdots$$

This equation has a fixed point at

$$b_* = \frac{8\pi^2}{9}\epsilon$$

which we can identify with the Wilson-Fisher fixed point. This fixed point is at small values of b near $d = 4$ and moves to large values of b as $(4-d)$ becomes of order 1.

At the Gaussian fixed point, the leading relevant operator is M^2, with $b_t = 2$. At the Wilson-Fisher fixed point, this operator direction is still relevant, but the eigenvalue b_t can be modified by terms of order b. We can evalute that correction using a method similar to that in the previous paragraph. We should do perturbation theory in both a and b, looking for term of order ab. This is given by

$$e^{-\int d^dx\,\left(\frac{1}{2}(\nabla M)^2 + \frac{at}{2}M^2 + \frac{b}{4}M^4 + \cdots\right)}$$

$$= e^{-\int d^dx\,\frac{1}{2}(\nabla M)^2}\left[1 + \cdots + \frac{at}{2}\frac{b}{4}\int d^dx\, M^4(x)\int d^dy\, M^2(y) + \cdots\right]$$

There is also a contribution to the operator M^2 that is of order b with no factors of a. This shifts the location of the fixed point by does not affect the stability behavior.

We can evaluate this expression just as we did above. I will consider both factors of M in the a term and two factors of M in the b term to be at large momentum. Then this term becomes

$$\frac{at}{2}\cdot\frac{b}{4}\cdot 6\int d^dx\; M^2(x)\int d^dy\;\langle M^2(x)M^2(y)\rangle$$

Again, the expectation value breaks up into two 2-point correlation functions.

$$=\frac{3at\cdot b}{4}\cdot 2\cdot\int d^dx\; M^2(x)\int d^4y\;\langle M(y)M(0)\rangle^2$$

This gives the same integral as we found above. The final result is

$$=\frac{1}{2}\cdot at\cdot b\cdot\frac{3}{8\pi^2}\log\lambda\int d^dx\; M^2(x)$$

This can be exponentiated

$$1+\frac{1}{2}at\cdot b\,\frac{3}{8\pi^2}\log\lambda\int d^dx\,M^2 = e^{-\int d^dx\left(-\frac{1}{2}at\cdot b\frac{3}{8\pi^2}\log\lambda\,M^2\right)}$$

Finally, we find a correction to the RG rescaling of (at),

$$at\rightarrow at\left(1+2\log\lambda-\frac{3b}{8\pi^2}\log\lambda+\dots\right)$$

We thus identify

$$b_t = 2-\frac{3b_*}{8\pi^2}+\mathcal{O}(b_*^2)$$

to be evaluated using the fixed-point value of b given above. Since $b_t = \nu^{-1}$, we find

$$\frac{1}{\nu} = 2 - \frac{1}{3}\epsilon + \mathcal{O}(\epsilon^2)$$

It is possible to carry out this same analysis for a Landau theory with an N-component order parameter. The result is

$$\frac{1}{\nu} = 2 - \frac{N+2}{N+8}\epsilon + \mathcal{O}(\epsilon^2)$$

In a similar way, one can compute the effect of b on the rescaling of the field M and obtain a correction to η. The first correction is second-order in b, so it is a little harder to obtain. The result is

$$\eta = \frac{(N+2)}{2(N+8)^2}\epsilon^2 + \cdots$$

It is interesting to compare these results to experimental determinations of the critical exponents for $N = 1, 2, 3$. The experimental values for ν are:

$$0.625(5), \quad 0.672(1), \quad 0.70(2)$$

The numbers in parentheses give the experimental errors in the last digit. Using the formula above, and naively setting $d = 3$ or $\epsilon = 1$, we find

$$0.60, \quad 0.63, \quad 0.65$$

This is already a dramatic improvement over the Landau theory result $\nu = 0.5$. Shortly after the Wilson-Fisher work, *Nickel* computed the b expansion of the critical exponents to quite high order in perturbation theory, and *Le Guillou* and *Zinn-Justin* applied some sophisticed techniques to resum the perturbation series. The result is the following set of predictions for ν in 3 dimensions:

$$0.630(2), \quad 0.674(6), \quad 0.711(8)$$

Thus, the Wilson-Fisher picture is in very good agreement with the known values of the critical exponents in magnetics and related systems with order-disorder transitions.