May 4

Landau's Theory of Phase Transitions

In the previous two lectures, we developed a great deal of formalism for the description of cooperative phenomena in statistical mechanics. In particular, through analysis with mean field theory, we found that systems with cooperative interactions can have a *critical point* which is the location of a transition from a *disordered* to an *ordered* phase. Many thermodynamics functions are singular at the critical point. In this lecture, I will present a semiquantitative method due to Landau that gives an intuitive and very illuminating picture of the critical point.

To begin this discussion, I will go back to mean field theory and discuss a little further the mean field theory solution of the Ising model. In our previous discussion, we found, in some cases, multiple solutions of the self-consistency equation of mean field theory. I claimed that certain solution were prefered over others because they corresponded to states of lower free energy. I will now explain that statement.

Mean field theory is an approximation scheme for computing the partition function and correlation functions of a statistical mechanics model. I will now show that it can be derived as a *variational principle*. This argument is not necessarily a more convincing for the validity of mean field theory, but it will give us some additional insight into the results of mean field theory calculations.

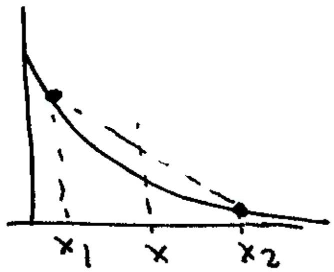
Consider, then, the calculation of

$$Z = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}$$

Usually, we cannot compute Z exactly, so we introduced some approximation. One way to do this is to find a Hamiltonian H_0 for which we can compute the partition function and then compute Z from the canonical ensemble for H_0. Thus, let Z_0 and F_0 be defined by

$$Z_0 = \sum_{\{s_i\}} e^{-\beta H_0(\{s_i\})}$$

It is a property of the exponential function that it is *convex* downward. That is, given x_1, x_2, and $x = \alpha x_1 + (1-\alpha)x_2$, with $0 \le \alpha \le 1$, then



$$\alpha e^{-x_1} + (1-\alpha) e^{-x_2} \ge e^{-x}$$

More generally, given $\{x_i\}$ and $\langle x_i \rangle = \sum_i p_i x_i$, with p_i probabilities such that $\sum_i p_i = 1$, then

$$\langle e^{-x_i} \rangle \ge e^{-\langle x_i \rangle}$$

It follows that

$$Z = \sum_{\{s_i\}} e^{-\beta H} = \sum_{\{s_i\}} e^{-\beta H_0} e^{-\beta(H-H_0)}$$

$$= Z_0 \langle e^{-\beta(H-H_0)} \rangle_0 \ge Z_0 e^{-\beta \langle H-H_0 \rangle_0}$$

where $\langle\rangle_0$ denotes an average in the canonical ensemble for H_0. Then

$$F \le F_0 + \langle H - H_0 \rangle_0$$

If H_0 has parameters, we can vary them to try to minimize the right-hand side. That will provide the best variational estimate of F.

For the Ising model on a d dimensional cubic lattice,

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - \mu h \sum_i s_i$$

To approximate this model, choose

$$\mathcal{H}_0 = -\eta \sum_i s_i$$

and let η be a variational parameter. It is easy to compute

$$Z_0 = \sum_{\{s_i\}} e^{-\beta \mathcal{H}_0} = \prod_i \left(e^{\beta\eta} + e^{-\beta\eta} \right) = [2\cosh\beta\eta]^N$$

and

$$\langle s_i \rangle_0 = \tanh \beta\eta$$

The expectation value of $\langle s_i \rangle_0$ is independent of i; I will write

$$\langle s \rangle \equiv \langle s_i \rangle_0$$

so that

$$\langle H - H_0 \rangle_0 = -dJN\langle s\rangle^2 - \mu h N \langle s \rangle + \eta N \langle s \rangle$$

Then the variational estimate is

$$F \leq -\frac{N}{\beta} \log(2\cosh\beta\eta) - NdJ\langle s \rangle^2 - N(\mu h - \eta)\langle s \rangle$$

Minimize the right-hand side with respect to η,

$$0 = \frac{\partial F(\eta)}{\partial \eta} = -N \tanh \beta\eta + N \langle s \rangle$$

$$+ \frac{\partial \langle s \rangle}{\partial \eta} \cdot N \cdot \left[-2dJ \langle s \rangle - \mu h + \eta \right]$$

On the right-hand side, the first line vanishes, because

$$\langle s \rangle = \tanh \beta\eta$$

Since $\partial \langle s \rangle / \partial \eta > 0$, we then find

$$\eta = 2dJ \langle s \rangle + \mu h$$

Putting this result into the formula for $\langle s \rangle$, we find the equation

$$\langle s \rangle = \tanh \beta \left(2dJ \langle s \rangle + \mu h \right)$$

which is just the self-consistency equation of mean field theory from our earlier treatment.

The structure of F resembles a Legendre transformation. I will now adjust the notation slightly to make this connection explicit. Change the normalization of M from that used in the previous lecture to

$$M = \mu \sum_i s_i$$

including the factor of μ. Then

$$M = \frac{1}{\beta}\frac{\partial}{\partial h}\log Z \qquad \text{or} \qquad M = -\left.\frac{\partial F}{\partial h}\right|_T$$

Now define a magnetic version of the Gibbs free energy as a Legendre transformation of $F(h)$,

$$G = F + Mh$$

This Gibbs free energy satisfies

$$\left.\frac{\partial G}{\partial M}\right|_T = h + \frac{\partial h}{\partial M}\left(\frac{\partial F}{\partial h} + M\right) = h$$

and can be though of as implementing the change of variables from h to M. At zero field, the thermodynamic states are

$$\left.\frac{\partial G}{\partial M}\right|_T = 0$$

In fact, since $G = F$ at $h = 0$ and lower values of F correspond to more probable thermodynamic states, the stable thermodynamic states correspond to the *minima* of $G(M)$. Another way to see this is to compute

$$\frac{\partial^2 G}{\partial M^2} = \frac{\partial h}{\partial M} = \frac{1}{\chi}$$

For stability, we must have

$$\chi = \frac{\partial M}{\partial h} > 0$$

so only a minimum of $G(M)$ is stable with respect to small fluctuation. The most probable thermodynamic state at zero field is the *global* minimum of $G(M)$.

The mean field theory estimate of $F(h)$ Legendre transforms to the following expression for $G(M)$:

$$\begin{aligned} G(M) &= F(\eta) + N\mu \langle s \rangle \cdot h \\ &= -\frac{N}{\beta} \log(2\cosh \beta\eta) - NdJ \langle s \rangle^2 + N\eta \langle s \rangle \end{aligned}$$

where

$$\langle s \rangle = \frac{M}{N\mu}$$

and η is given in terms of $\langle s \rangle$ by

$$\eta = 2dJ\langle s \rangle + \mu h$$

I claim that the equation

$$\frac{\partial G}{\partial M} = h$$

should give us back the variational or self-consistency equation. Explicitly,

$$h = \frac{\partial G}{\partial M} = \frac{1}{N\mu}\frac{\partial G}{\partial \langle s \rangle} = (-2dJ\langle s \rangle + \eta)/\mu$$

$$+ \frac{1}{\mu}\frac{\partial \eta}{\partial \langle s \rangle}(-\tanh \beta\eta + \langle s \rangle)$$

The second line is zero by the relation above for η, and the first line is then precisely the self-consistency equation.

It is straightforward to construct $G(M)$ explicitly from the mean field theory formulae. Eliminating the η in favor of $\langle s \rangle$ implies that

$$h = \frac{1}{\mu}(\eta - 2dJ\langle s \rangle)$$

so that the equation $\partial G/\partial M = h$ becomes

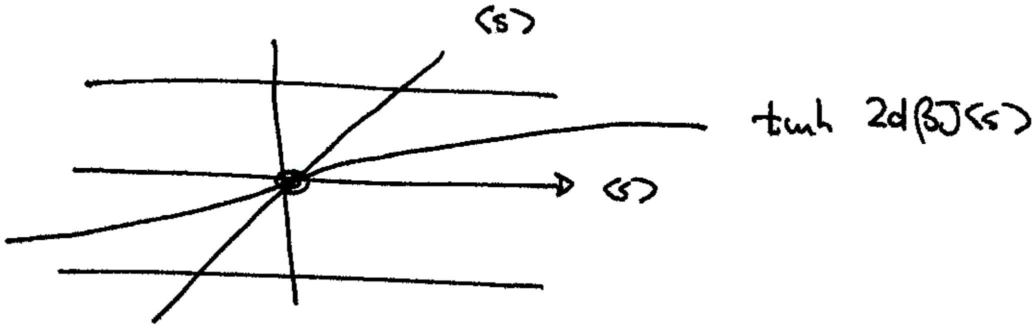
$$\frac{\partial G}{\partial \langle s \rangle} = \frac{N}{\beta}(\tanh^{-1}\langle s \rangle - 2dJ\beta\langle s \rangle)$$

The right-hand side of this equation is not so transparent, but we can make sense of it in relation to the graphical solution of the self-consistency equation discussed previously. I will first make the observations that this expression is odd in $\langle s \rangle$, behaves near $\langle s \rangle = 0$ as

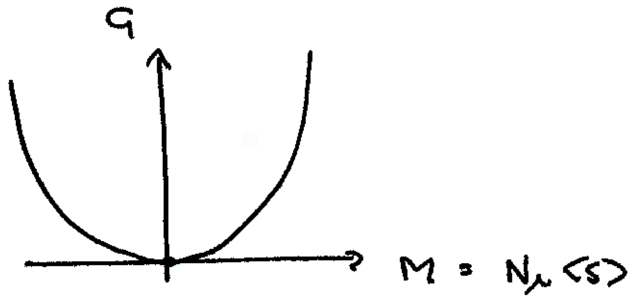
$$\frac{N}{\beta} \cdot \langle s \rangle \cdot (1 - 2d\beta J) + \mathcal{O}(\langle s \rangle^3)$$

and, as $\langle s \rangle \to \pm 1$, tends to $\pm\infty$.

For $T > T_c$ or $2d\beta J < 2d\beta_c J = 1$, $\partial G/\partial M$ is positive for $\langle s \rangle > 0$, corresponding to the situation in the graphical solution

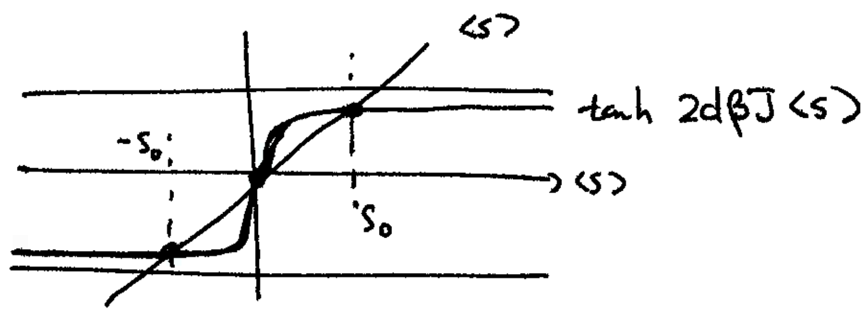


The integral of this expression, $G(M)$, then increases monotonically away from $M = 0$ in either direction.

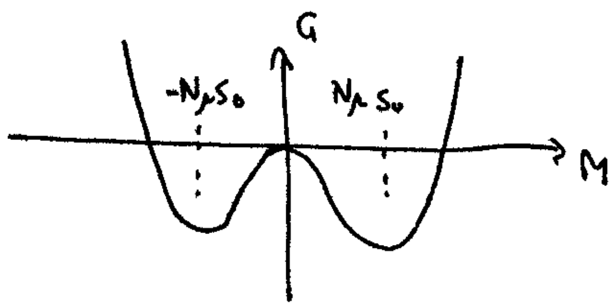


There is a unique minimum of $G(M)$, corresponding to zero magnetization for $h = 0$.

For $T < T_c$ or or $2d\beta J > 2d\beta_c J = 1$, $\partial G/\partial M$ is negative for $\langle s\rangle > 0$ and sufficiently small, though at large values of $\langle s\rangle$, $\partial G/\partial M$ must become positive. This behavior corresponds to the situation in the graphical solution



The integral $G(M)$. then decreases to a minimum and, after that, increases monotonically.



The form of $G(M)$ must be symmetric under $\langle s \rangle \leftrightarrow -\langle s \rangle$. Thus $G(M)$ has a pair of minima at symmetric points $\pm M_0$ and at the same depth, located at

$$\pm M_0 = \pm N\mu\, s_0$$

where $\langle s \rangle = \pm s_0$ are the two nontrivial solutions of the self-consistency equation. The solution of the self-consistency equation at $\langle s \rangle = 0$ is seen to be a *maximum* of $G(M)$ and therefore does not correspond to a stable thermodynamic state.

The Gibbs free energy $G(M)$ gives us a great deal of insight into the thermodynamics of magnetic ordering. This function contains the magnetization directly. It has all of the symmetries of the original thermodynamic problem. However, it can have minima that do not respect these symmetries. An asymmetric minimum of $G(M)$ is a stable but non-symmetric thermodynamic state. When such a point is a global minimum of $G(M)$, we have *spontaneous symmetry breaking.*

If $G(M)$ has a minimum at an asymmetrical location, it must have other minima at other locations related to this one by symmetry. In the example above, if $G(M)$ has a minimum at $M = +M_0$, it must also have a degenerate minimum at $M = -M_0$. These two states correspond to the two partial canonical ensembles that we discussed in relation to the transfer matrix. The free energies of these states are given by the two degenerate largest eigenvalues of the transfer matrix. Specifics of the boundary condition or of the method of preparation of the state determine which state is realized in a given physical situation.

This analysis suggests a general strategy for building a phenomenological picture of system whose thermodynamics contains the possibility of spontaneous symmetry breaking.

1. Determine the symmetry of the model. This is described by a symmetry group $\mathbf{G}$. Find a thermodynamic observable that transforms under this symmetry. That observable is called the *order parameter.*

   For an Ising ferromagnetic, the symmetry group $\mathbf{G}$ is the group of transformations $s \to s$, $s \to -s$. This group is called Z_2. Its nontrivial element is parity P. The order parameter is M, with $P : M \to -M$.

2. Write a general function G depending on the order parameter and symmetric under $\mathbf{G}$. Associate parameters in $G(M)$ with the temperature and other thermodynamic parameters.

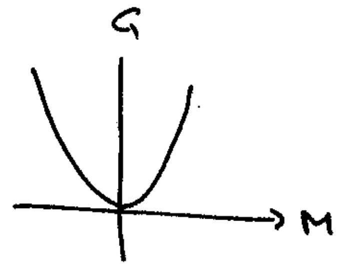
3. Varying these parameters, solve for the minima of G.

In the vicinity of a critical point describing the onset of ordering, the value of the order parameter will be small. Then it might make sense to expand G in a Taylor expansion in the order parameter. This approach was laid out and explored by Lev Landau begining in 1937. The resulting expression for the Gibbs free energy G is called the *Landau effective free energy.*

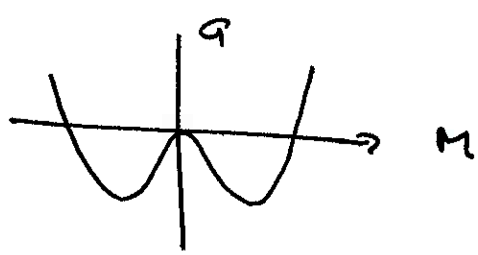
I will now carry out this analysis for the Ising model. The Gibbs free energy $G(M)$ must be symmetric under $M \to -M$. Then

$$G(M) = \frac{1}{2} A M^2 + \frac{1}{4} b M^4 + \cdots$$

I will assume that $b > 0$ for stability, but I will consider both possible signs for A. For $A > 0$, $G(M)$ has the form



But for $A < 0$, $G(M)$ has the form



with asymmetric minima. The point of transition between these two behaviors is the critical point. Again in the spirit of a Taylor expansion, write

$$A = a(T - T_c)$$

with a a positive constant. Then the minima of $G(M)$ are given by the solutions of

$$\frac{\partial G}{\partial M} = 0 = a(T-T_c)M + bM^3$$

If $T > T_c$, there is only one solution at $M = 0$. If $T < T_c$, there are two new solutions, at the points

$$M = \pm\left[\frac{a}{b}(T_c - T)\right]^{1/2}$$

and these are the stable minima of $G(M)$.

In a nonzero magnetic field, the thermodynamic state is the solution of

$$\frac{\partial G}{\partial M} = h = a(T-T_c)M + bM^3$$

Above T_c, for small h,

$$M \cong \frac{h}{a(T-T_c)}$$

Then

$$\chi = \frac{\partial M}{\partial h} = \frac{1}{a(T-T_c)}$$

Just at $T = T_c$,

$$h = bM^3 \quad \text{or} \quad M \sim h^{1/3}$$

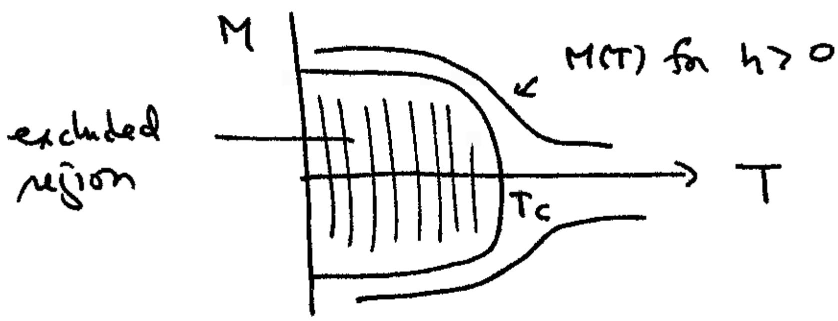
We have now recovered the major thermodynamic predictions of mean field theory. There exists a critical point at $T = T_c$, $h = 0$, with a nonzero magnetization appearing for $T < T_c$. In the vicinity of the critical point, we find the singular behaviors,

$$M \sim (T_c - T)^{1/2} \qquad \chi \sim |T - T_c|^{-1} \qquad M \sim h^{1/3}$$

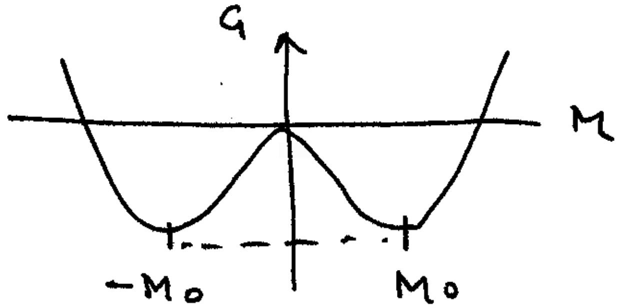
$$\text{at } h = 0,\ T < T_c \qquad h = 0 \qquad T = T_c$$

We know that these power laws are not correct for the 2 dimensional Ising model, so there must be something wrong with Landau theory. I will discuss this issue later in the course. For the moment, I will exploit this very useful theory to learn more about the behavior of thermodyamic system in the neighborhood of their crtiical points.

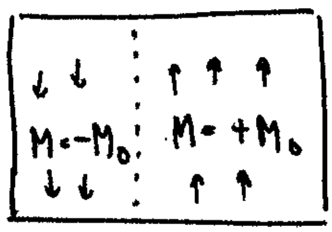
For example, Landau theory gives us an insight into the strangest feature of the phase diagram that we found in mean theory. In the plane of M versus T, we found an *excluded region* that did not correspond to any thermodynamic state.



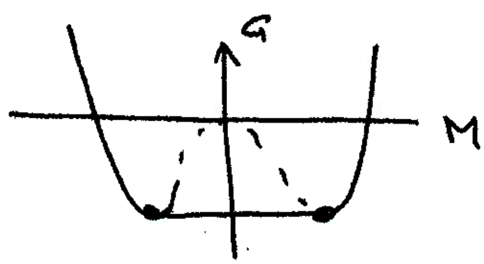
In terms of $G(M)$, the excluded region corresponds to the region between the two minima



If M is in this region, we could in principle try to build a uniform thermodynamic state with $G(M)$ higher than the minimum. However, a better solution would be to build a state with *phase separation*



so that $M = +M_0$ in some part of the volume and $M = -M_0$ in the rest of the volume, with the sizes of these regions adjusted so that the *average* magnetization is the given M. Any such state will have $G = G(M_0)$, and these states continuously interpolate from the minimum at $M = M_0$ to the minimum at $M = -M_0$. The correct form of the Gibbs free energy is then not the analytic function $G(M)$ written above but rather a non-analytic function of the form



with a *flat* segment crossing the excluded region.

The Landau phenomenological theory gives the same function $G(M)$ for all systems with a given symmetry group **G**. There is then some logic in grouping together systems with different kinds of phase transitions that share a common symmetry group. This classification turns out to be useful even for those quantities for which Landau theory does not give the correct results. It turns out that systems in the same class have the same behavior in the vicinity of the critical point.

Ferromagnets fall into classes according to whether the atomic spins are constrained by their crystal environment to fluctuate along an axis or in a preferred plane or are allowed to fluctuate isotropically. In these cases,

- In *Ising* ferromagnets, the spins fluctuate along a preferred axis. The symmetry group **G** is the group Z_2 represented by $(1, P)$. The order parameter M has one component.

- In XY ferromagnets, the spins fluctuate in a preferred plane. The symmetry group **G** is the group of rotations in 2 dimensions. The order parameter can be represented as a vector $\vec{M}$ with two components, or, alternatively, as a complex number $M = M^1 + iM^2$.

- In *Heisenberg* ferromagnets, the spins fluctuate isotropically in 3 dimensions. The symmetry group **G** is the group of rotations in 3 dimensions. The order parameter can be represented by a vector $\vec{M}$ with three components.

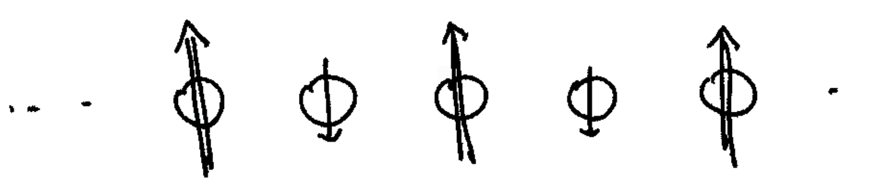
You can see how to generalize Landau theory so that the symmetry group **G** is a larger group of rotations and $\vec{M}$ is an N-component vector. You might think that these Landau theories would not describe any real system in 3 dimensions. But wait, please.

In some materials with transition-metal atoms, neighboring spins prefer to be antiparallel,

In an ordered phase of this type, the magnetization is zero, but the *staggered magnetization*

$$M = \mu \sum_i (-1)^i s_i$$

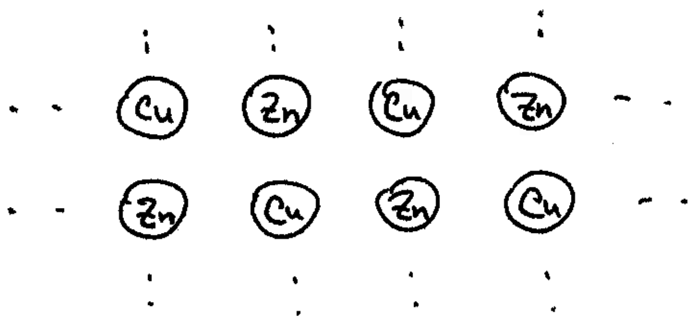
where $(-1)^i$ equals 1 on even sites of the lattice and -1 on odd sites, can have a thermodynamic expectation value. Then M_{st} is the order parameter. Such a systems is called an *antiferromagnet.* It does not show anomalies in χ, but its specific heat is singular at a temperature called the *Néel temperature*, corresponding to a phase transition to a state with $M_{st} \neq 0$. The staggered spin ordering can be confirmed by neutron scattering. More complex magnetic order, such as *ferrimagnetism*



is also seen in some materials. Some examples of these behaviors are

- Ferromagnets: Fe, Co, Ni, Cu_2MnAl
- Antiferromagnets: MnO, FeO, CoO
- Ferrimagnets: Fe_3O_4 (magnetite), $CoFe_2O_4$

Another type of ordering occurs in *binary alloys.* An example is β-brass, which is a 1:1 alloy of Cu and Zn. In the ordered state, atoms of the two different types occupy even and odd lattice sites



We can represent the Cu and Zn positions in the crystal lattice by defining a spin variable

$$S_i = \begin{cases} +1 & \text{Cu} \\ -1 & \text{Zn} \end{cases}$$

Then, this is a system with Z_2 symmetry whose order parameter is the staggered magnetization

$$M = \sum_i (-1)^i S_i$$

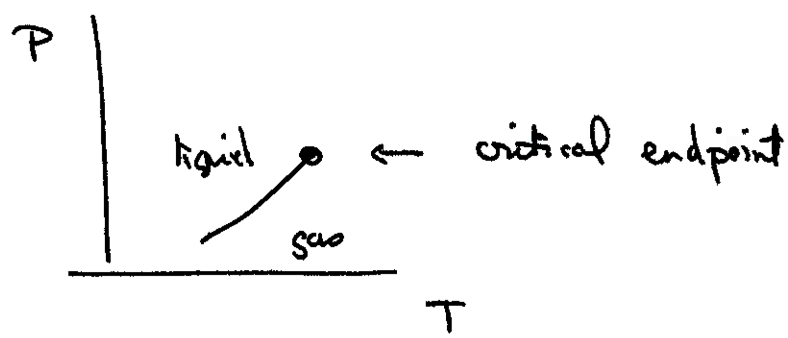
The Landau free energy for β-brass is *the same* as that for the Ising model.

In a *superfluid*, as I will describe later, the order parameter is a quantum mechanical wavefunction occupied by a macroscopic number of (Bose-Einstein) particles. A wavefunction is a complex number, and quantum mechanics has the symmetry of phase rotation

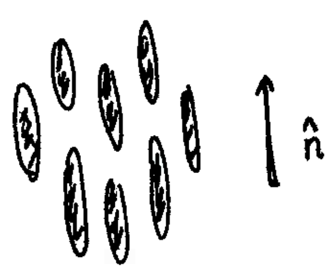
$$\psi(x) \rightarrow e^{i\alpha}\psi(x)$$

which is isomorphic to the symmetry group of 2-dimensional rotations. The Landau theory of a superfluid is then the same as for an XY ferromagnet.

The liquid-gas transition is more subtle. There is no symmetry that connects the liquid and gas phases. We can consider the *density* to be the order parameter that distinguishes the two phases. Typically, the liquid-gas transition is *discontinuous*, with a finite change in the density from one phase to the other. However, usually the phase diagram includes a *critical endpoint* where the difference between liquid and gas disappears. This endpoint is described by the Landau free energy of the Ising model. I will discuss this system in some detail in a later lecture.



Liquid crystals are systems with large molecules that can have orientational order. In a *nematic* liquid crystal, the order parameter is the average orientation vector $\hat{n}$ of the molecules



If the molecules do not have definite ends, there is no difference between the orientations described by $\hat{n}$ and $-\hat{n}$. This affects some aspects of the ordering, as I will discuss in the next lecture.

There are even more exotic situations. In general, the order parameter is a vector M^a in an irreducible representation of the symmetry group $\mathbf{G}$. Then

$$G(M) = \frac{1}{2}a(T-T_c)(M^a)^2 + \cdots$$

The symmetry **G** might allow several quartic terms, depending on the 4-index invariants allowed by that representation. There are cases in which the dimension of the representation is 4, 5, or higher. These realize Landau theories with more than 3 components in the order parameter.

In addition to obtaining information about global thermodynamic quantities from Landau theory, we can use Landau theory to discuss situations in which the order parameter changes slowly with position. To do this, consider the order parameter to have a local value $M(x)$ and the Landau polynomial function as a *local density* of free energy. The global Gibbs free energy is then the *spatial integral* of this density. We should still consider this description as coarse-grained with respect to the atomic distance scale. If the order parameter changes from one point to another, there should be a positive free energy cost to establish the nonuniform configuration. Then the full phenomenological Gibbs free energy would read

$$G = \int d^3x \left\{ \frac{1}{2}\rho(\vec{\nabla} M)^2 + \frac{1}{2}a(T-T_c)M^2 + \frac{1}{4}bM^4 \right\}$$

with ρ, a, b positive constants. The notation $G[M]$ denotes the fact that G is a *functional* of the function $M(x)$. I will now study some consequences of this formula in the Ising case where $M(x)$ has one component.

A stable thermodynamic state is a minimum of $G[M]$ with respect to the function $M(x)$. Usually, $M(x)$ will be constrained to satisfy some boundary conditions. Then, away from the boundaries, $M(x)$ must satisfy the variational equation

$$\delta G = 0$$

The variation of G above is given by

$$\delta G = \int d^3x \left\{ \tfrac{1}{2}\rho \cdot 2 \cdot \vec{\nabla}\delta M \cdot \vec{\nabla} M + a(T-T_c)\,\delta M \cdot M + b\,\delta M\, M^3 \right\}$$

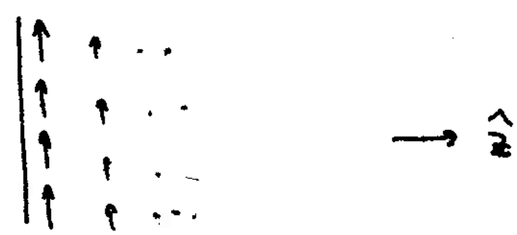
If $M(x)$ is fixed on the boundary, we can integrate by parts and write this expression as

$$\delta G = \int d^3x \;\; \delta M \left\{ -\rho\nabla^2 M + a(T-T_c)M + b M^3 \right\}$$

Then $M(x)$ obeys the partial differential equation

$$\nabla^2 M = \frac{a}{\rho}(T-T_c)\,M + \frac{b}{\rho} M^3$$

We can use this equation to revisit some of the problems that we studied for the 1 dimensional Ising model. First, consider the situation in the unmagnetized state at $T > T_C$ in which we pin the magnetization density on the boundary to a definite value M_0. With this boundary condition, we can solve for $M(x)$ in the interior.



This is a 1-dimensional problem, so the partial differential equation reduces to an ordinary differential equation. If $M(x)$ is small, we can ignore the M^3 term; then we need to solve

$$\frac{d^2}{dz^2} M = \frac{a(T-T_c)}{e} M$$

The solution, with M_0 on the boundary at $z = 0$, is

$$M(z) = M_0 \exp\left[- \frac{a(T-T_c)}{e} z \right]$$

If M_0 is large, this is still the correct behavior for large z, with a different overall normalization. We see that $M(z)$ falls *exponentially*,

$$M(z) \sim (\text{const}) \cdot e^{-z/\xi(T)}$$

with the correlation length

$$\xi(T) = \left[\frac{e}{a(T-T_c)} \right]^{\frac{1}{2}} \sim (T-T_c)^{-\frac{1}{2}}$$

The correlation length tend to infinity as $T \to T_C^+$ as a non-integer power of $|T - T_C|$.

It is not difficult to set up and solve a 3-dimensional version of this problem. Imagine a small region in which $M(x)$ is fixed to a positive value. Then we can use the variational equation to compute how $M(x)$ falls off as we move away from this region. For convenience, I will put the center of the region in which $M(x)$ is fixed at $\vec{x} = 0$ and consider the fixed region to be symmetric about $\vec{x} = 0$. Then we must solve

$$\nabla^2 M + \frac{a(T-T_c)}{e} M = 0$$

for a solution symmetric about $\vec{x} = 0$. At $T = T_C$, this is just the Laplace equation. The symmetric solution is

$$M(x) = \frac{c}{4\pi} \frac{1}{|\vec{x}|}$$

a Coulomb potential. The generalization to d dimensions also gives a Coulomb potential, now of the form

$$M(x) \sim \frac{1}{|\vec{x}|^{d-2}}$$

For $T > T_C$, the equation is that of a *Yukawa potential.* The spherically symmetric solution in 3 dimensions is

$$M(x) = \frac{c}{4\pi} \frac{1}{|\vec{x}|} e^{-\left[\frac{a(T-T_c)}{\rho}\right]^{\frac{1}{2}} |\vec{x}|}$$

This solution should also describe the long-distance behavior of the spin-spin or magnetization-magnetization correlation function

$$\langle M(\vec{x}) \, M(\vec{y}) \rangle$$

since this function is constrained to be large and positive at $\vec{x} = \vec{y}$ and to fall off symmetrically as these points are separated. We see, then, that

$$\langle M(\vec{x}) \, M(\vec{y}) \rangle \sim (\text{const}) \cdot \frac{1}{|x-y|} e^{-|\vec{x}-\vec{y}|/\xi(T)}$$

where the correlation length $\xi(T)$ is given by the same expression as is quoted above.

It is instructive to repeat this exercise for $T < T_C$. Now the stable thermodynamic state is one of the minima of $G(M)$ at $M = M_0$ or $M = -M_0$. Choose the first of these for definiteness,

$$M = + \left[\frac{a(T_c - T)}{b}\right]^{\frac{1}{2}} = M_0$$

In the expression for $G[M]$, write

$$M(x) = M_0 + m(x)$$

The variational equation becomes

$$\nabla^2 m(x) = -\frac{a(T_c - T)}{\rho}(M_0 + m) + \frac{b}{\rho}(M_0 + m)^3$$

Away from regions where $M(x)$ is constrained, the deviation $m(x)$ should be small. Then, expand to linear order in $m(x)$. This gives the equation

$$\nabla^2 m = -\frac{a(T_c - T)}{\rho} M_0 + \frac{b}{\rho} M_0^3$$
$$- a\left(\frac{T_c - T}{\rho}\right) m + 3\frac{b}{\rho} M_0^2 m + \cdots$$

The first line vanishes when we use the explicit formula for M_0. The equation then simplifies to

$$\nabla^2 m = +2\,\frac{a(T_c - T)}{\rho}\, m$$

This equation thus also takes the form of the equation for the Yukawa potential.

Using this equation, we can repeat the analyses done above for $T > T_C$. In the magnetized state, we can consider a boundary condition on $M(x)$ is which $M(x)$ is pinned to a large value on a wall at $z = 0$. Then, far from the wall,

$$M(x) = M_0 + (\text{const}) \cdot e^{-z/\xi(T)}$$

with

$$\xi(T) = \left[\frac{2a(T_c - T)}{\rho}\right]^{-\frac{1}{2}}$$

Similarly, the asymptotic behavior of the magnetization-magnetization correlation function has as its asymptotic behavior

$$\langle M(x) M(y) \rangle \sim M_0^2 + \frac{(\text{const})}{|x-y|} \cdot e^{-|x-y|/\xi(T)}$$

with this same value of $\xi(T)$. Notice that $\xi(T)$ here has an extra factor of $\sqrt{2}$ relative to the value in in the case $T > T_C$, but it still diverges as $|T - T_C|^{-1/2}$ as $T \to T_C^-$.

There is another situation in which the variational equation is very useful. Consider a situation with $T < T_C$ in which we set up boundary conditions so that $M(x) \to M_0$ as $z \to \infty$ but $M(x) \to -M_0$ as $z \to -\infty$. The resulting configuration $M(x)$ would then give the shape of the *domain wall* that forms the boundary between the regions of up and down magnetization. To find $M(x)$ in this case, we need to solve the full nonlinear equation

$$\frac{d^2}{dz^2} M(z) = -\frac{a(T_c - T)}{\rho} M + \frac{b}{\rho} M^3$$

for $M(z)$, a function varying in one dimension only. A convenient ansatz is

$$M(z) = \left[\frac{a(T_c - T)}{b}\right]^{\frac{1}{2}} f(w)$$

where $f(w)$ is a function of the variable

$$w = \left[\frac{a(T_c - T)}{2\rho}\right]^{\frac{1}{2}} z$$

Plugging this formula in to the variational equation, all of the coefficients scale out, and we are left with the equation

$$\frac{1}{2}\frac{d^2}{dw^2} f(w) = -f(w) + f^3(w)$$

where $f(w) \to 1$ as $w \to \infty$, $f(w) \to -1$ as $w \to -\infty$. A solution of the differential equation is

$$f(w) = \tanh w$$

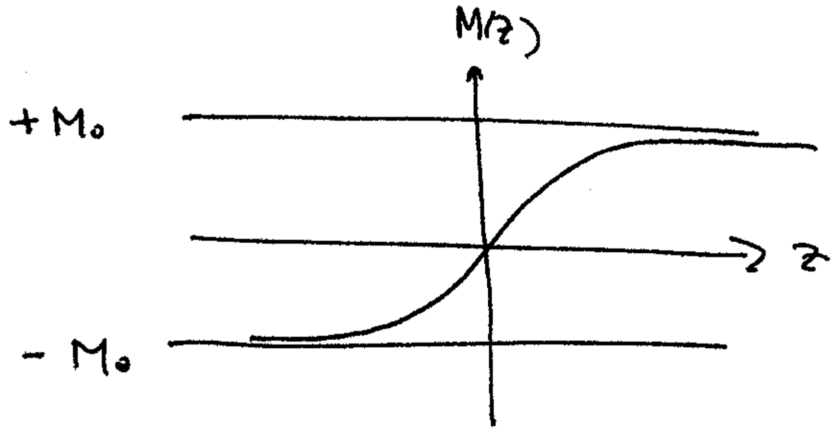
since

$$\frac{d^2}{dw^2}\tanh w = -2\frac{\sinh w}{\cosh^3 w} = -2\left[\tanh w - \tanh^3 w\right]$$

This function satisfies the boundary conditions as well. Then the solution to the original problem is

$$M(z) = M_0 \tanh\left(\left[\frac{a(T_c - T)}{2\rho}\right]^{\frac{1}{2}} z\right)$$

which provides a smooth transition between the two phases,



Since

$$\tanh w = \frac{1-e^{-2w}}{1+e^{-2w}} = 1 - 2e^{-2w} + \cdots$$

the domain wall solution behaves asymptotically like

$$M(z) \sim M_0 - (\text{const}) \cdot e^{-\left(\frac{2a(T_c-T)}{\rho}\right)^{\frac{1}{2}} z}$$

in agreement with our earlier results. The center of the solution can be moved to an arbitrary position. Replacing the coefficient of z by the correlation length for $T < T_C$, we find finally the general form of a domain wall

$$M(z) = M_0 \tanh\left[\frac{z - z_0}{2\xi(T)}\right]$$