Physics 212 – Final Exam

This exam is due at noon on Tuesday, June 8. Hand it in to the TA Daniele Alves in Varian 363. If you have any questions about the exam, please contact Michael Peskin by email at the address mpeskin@slac.stanford.edu. If errata are reported, he will announce them on the course Web page.

Do not collaborate on this exam. Return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Alternatively, copy the Stanford honor code from

http://www.stanford.edu/dept/vpsa/judicialaffairs/guiding/pdf/honorcode.pdf ,

sign it, and attach it to your exam solution.

The exam is worth a total of 100 points.

The exam is open-book. However, if you make strong use of a reference other than the class textbooks and notes, please cite the reference in your solutions. The exam can be worked using the course materials only. You will learn more if you do not try to find the answers in the literature.

1. (50 points) Set up a Carnot cycle using an inflatable empty balloon heated to very high temperature. Yes, I mean very high, a temperature of approximately 200 MeV. Such inflatable balloons are easy to build only on final exams. You may ignore the mass of the electron ($m_e c^2 = 0.51$ MeV) in this problem. You may ignore muons and heavier elementary particles.

   (a) Consider the properties of a spherical balloon of radius 1 m at a temperature of 200 MeV. If the balloon would be truly empty (vacuum inside) at zero temperature, then at a temperature of 200 MeV it will contain photons, electrons, and positrons. Compute the number of each. What is the pressure on the walls?

   (b) Take a balloon of negligible radius out of your pocket. Heat it to a temperature of 200 MeV, still at negligible radius. Then let the balloon expand at this temperature to a radius of 1 meter. How much energy is absorbed by the balloon?

   (c) Now let the balloon expand adiabatically to a radius of 10 m. What is the final temperature of the balloon? How much work was done in this process?

   (d) How fast can we let the balloon expand and still have the process be adiabatic? The number of reactions $e^+e^- \to \gamma\gamma$ per cm^3 per second is given approximately by

   $$n(e^-)n(e^+) \cdot \sigma \cdot c$$

   where c is the speed of light and σ is the e^+e^- annihilation cross section, $\sigma \approx 10^{-24}$ cm^2.

(e) Estimate the mean free path for photons during the adiabatic stage. Notice that the matter in the balloon behaves like a fluid. For a fluid, the speed of sound c_s is given by

$$c_s = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} ,$$

in adiabatic compression or expansion. Compute the speed of sound in this fluid.

(f) Squeeze the balloon down to negligible size in an isothermal process at the final temperature. How much work must be done on the balloon to do this?

(g) Discuss the fourth leg of the Carnot cycle.

(h) Compute the efficiency of the Carnot cycle. Does this agree with Carnot's general result?

2. (50 points) Consider a *ferrimagnetic* material with alternating atoms A and B with Ising spins S_{Ai}, S_{Bi}. Let the coupling between A and B be antiferromagnetic with strength J. For a one-dimensional system,

$$+J \sum_i S_{Ai}(S_{B(i-1)} + S_{Bi})$$

Introduce also an antiferromagnetic coupling between nearest-neighbor A atoms. In one-dimension, this part of the Hamiltonian reads

$$+K \sum_i S_{Ai} S_{A(i+1)}$$

Do not include any direct coupling between the B atoms; this would make the problem even more complicated. In this problem, assume that J and K are both positive. Please take the external field h to be zero.

(a) Consider first a 1-dimensional chain with alternating A and B atoms. Find the free energy of this model exactly for $K = 0$ and for $J = 0$. Show that $\langle S_{Ai} \rangle = 0$ at any $T > 0$.

(b) Find the free energy of this model exactly as a function of J and K. both nonzero. It is most straightforward to set this up using a 4×4 transfer matrix.

(c) Show that, at zero temperature, if K is sufficiently small, this one-dimensional system has perfect ferromagnetic order at $T = 0$. However, if $K > 0$ and sufficiently large, the system has antiferronagnetic order at $T = 0$. Find the boundary between these two behaviors.

(d) Compute the spin-spin correlation function

$$\left\langle S_{Ai} \; S_{A(i+n)} \right\rangle$$

and discuss the behavior in the limits $J \gg K$ and $K \gg J$.

(e) Next, consider a 2-dimensional square lattice on which even and odd sites are occupied by A and B atoms. That is, the atoms are arranged like a checkerboard with A atoms on the red squares and B atoms on the black squares. Let adjacent A and B atoms have antiferromagnetic coupling of strength J. Let A atoms at the opposite corners of each elementary square (or, with the checkerboard analogy, on neighboring red squares) have antiferromagnetic coupling of strength K. Then all A atoms couple to four neighboring B atoms and to four neighboring A atoms. Assume first that the A atoms have ferromagnetic order (all A spins aligned in the same direction). Write down the mean field theory equations for $\langle S_{Ai} \rangle$ and $\langle S_{Bi} \rangle$.

(f) Use these equations to solve for the critical temperature as a function of J and K, assuming that A atoms have ferromagnetic order.

(g) Next, assume that the A atoms have antiferromagnetic order. Write the mean field theory equations and compute the critical temperature.

(h) Analyze this model at $T = 0$. Find the regions of J and K that give ferromagnetic and antiferromagnetic order for the A atoms.

(i) Sketch the phase diagram of the model in the plane of J and K.