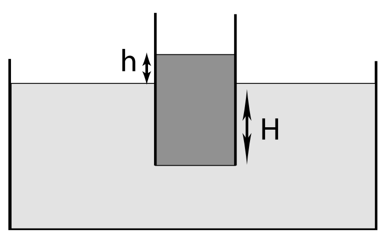
Physics 212 – Final Exam

The exam is worth a total of 100 points. The distribution of points is indicated below.

If you make strong use of a reference other than the class textbook and notes, please cite the reference in your solutions.

1. (30 points) In the system shown in the figure, a semipermeable membrane separates a tube containing a sugar solution from a large volume of water containing very little sugar. The osmotic pressure of the sugar acts to pull water into the column, so that the water in the column is higher than the water outside by a distance h.

   Assume that the membrane is rigid and permeable to water but impermeable to sugar. In working this problem, you can make a number of other assumptions, even though they are excessively simple: Assume that the sugar molecules in water are ideal and noninteracting, that the sugar adds negligibly to the mass/volume of the water, and that the water is incompressible. In the final situation shown in the figure, let the concentration of sugar in the water be n (molecules/cm^3), and let the concentration of sugar in the reservoir below be a fraction f of this value, and let the area of the tube be A.



   (a) Find the height h to which the water rises in the column. You can do this, for example, by minimizing the pressure on the membrane. You should obtain a formula that involves the density ρ of the water, the heights H and h, the concentration n, and other needed factors. Put in the numbers for a 1 mole/liter solution at room temperature and $H = 1$ cm. Assume that $f \ll 1$.

   (b) A molecular pump can move sugar from the reservoir into the column through the membrane. How much free energy is needed to move 1 molecule of sugar?

   (c) Imagine that the pump moves sugar molecule by molecule from below to above, raising the level of the water by dh. How much free energy is required? The work done on the water is $\rho g A h\, dh$. How does this compare to the free energy consumed by the pump?

2. (30 points)

   A simple model of neutron star matter is a Fermi gas of neutrons. Neutrons are spin-$\frac{1}{2}$ particles of mass m_n. Neutrons have some attractive interactions, but ignore these and treat the neutrons as non-relativistic free particles. Let n_0 be a typical nuclear density. In this problem, 'density' means number density, neutrons or quarks/cm^3.

   Quark matter is a Fermi gas of u, d, and s quarks. Each of these particles has spin $\frac{1}{2}$ and also carries a quantum number called *color* that takes 3 values (red, green, blue). Treat the quarks as massless, relativistic fermions (but do not include the antifermions). Model the interactions that bind quarks into neutrons by assigning quark matter a fixed positive energy B per unit volume.

   In comparing quark matter and neutron matter, assume that neutrons can dissociate into quarks, or vice versa, at the interface. Baryon number (or quark number) is conserved. A neutron contains 3 quarks. Assume that u, d, and s quarks can freely interconvert by emitting and absorbing electrons and neutrinos, which you can assume are present in large enough numbers with $\mu = 0$.

   (a) Write the formula for the pressure of each type of matter at zero temperature as a function of the number density. Adjust B so that quark matter and neutron matter coexist at the same pressure at zero temperature and at a density of $2n_0$ on the side of neutron matter. What is the corresponding quark matter number density? If the algebra gets too horrible, made judicious approximations.

   (b) If the pressure is increased, does this favor quark matter or neutron matter?

   (c) It would be good to compute the coexistence pressure as a function of temperature, and to and compute the densities at coexistence, for small but nonzero T. Set up this calculation, giving the relevant formulae for pressure and density.

3. (40 points) Consider the magnetic model on d-dimensional square lattice:

$$H = +J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i \tag{1}$$

   This is an antiferromagnet in a uniform magnetic field. Take the s_i to be Ising spins with values ± 1.

   (a) Find the ground state of this system at zero temperature as a function of the magnetic field strength h.

   (b) Analyze this system using mean field theory. Write the self-consistency equations for the order parameters. There are two order parameters, the magnetization $s = \langle s_i \rangle$ and the staggered magnetization $t = \langle (-1)^i s_i \rangle$. Both quantities should be involved.

(c) Find the critical temperature for $h = 0$. What happens to the phase transition when a small h is turned on? Find the formula for the transition temperature to order h^2.

(d) Sketch the phase diagram of this model.