May 20

Critical Exponents

At this point in the course, we have discussed all aspects of rthe phase diagram of a complex thermodynamic system except for one special region—the vicinity of the critical point. Here there are important questions that I have left unanswered. I have noted that mean field theory and Landau theory give an excellent qualitative picture of the phase diagrams of systems with cooperative interactions. They correctly predict the non-analytic behavior of thermodynamic functions at the critical point. However, these methods do not predict the correct power laws that characterize the singularities. In the final lectures of this course, I would like to present methods for understanding these power laws, called *critical exponents.*

The special properties of the vicinity of the critical point are the result of the long-range of correlations among thermal fluctuations. In the Landau theory for a ferromagnet, we studied the spin-spin correlation function. We saw that it typically falls off as exponentially as a function of distance

$$T > T_c \qquad \langle S(x) S(0) \rangle \sim \frac{A}{|\vec{x}|^a} e^{-|\vec{x}|/\xi(T)}$$

$$T < T_c \qquad \langle S(x) S(0) \rangle \sim \langle S(0) \rangle^2 + \frac{B}{|x|^a} e^{-|\vec{x}|/\xi(T)}$$

$$\text{as} \quad |\vec{x}| \to \infty$$

However, the *correlation length* $\xi(T)$ becomes infinite as $T \to T_C$. We can understand that the range of correlations must become infinite at T_C because the asymptotic behavior of the function changes there, from

$$\langle S(x) S(0) \rangle \longrightarrow 0$$

to

$$\langle S(x) S(0) \rangle \longrightarrow \langle S(0) \rangle^2 \neq 0$$

But this requirement brings special and quite strange implications.

Imagine that we have a magnet with $T > T_C$ and $h = 0$ and decrease T slowly. When T becomes less than T_C, we will have a magnetized state in which the magnetization points in a definite direction. However, above T_C, there is a unique thermodynamic state that can have no overall preferred direction for its magnetization. Clusters of spins form with a preferred direction of spin up. However, these clusters must be embedded in larger clusters which could equally well have a preferred direction of spin down. As $T \to T_C$, the size of these clusters of correlated spins becomes larger and larger until, as T crosses T_C, the largest correlated cluster is one that fills the whole sample and chooses its overall orientation.

At $T = T_C$, we have relevant fluctations at all distance scales. The simplest hypothesis for the description of these fluctuations is that their amplitude is precisely *scale-invariant.* That is, the pattern of clustering looks the same on *any* distance scale much greater than that of the original lattice.

The simplest expression of scale-invariance is in the behavior of the spin-spin correlation function. Let

$$C(x) = \langle s(x)\, s(0) \rangle$$

If the theory is scale-invariant, the correlation function should be unchanged if we rescale distances by a factor λ and also, possibly, rescale the lengths of spins by a factor of the λ^A. More precisely,

$$C(x) = \lambda^{-2A} \langle s(x/\lambda)\, s(0) \rangle = \lambda^{-2A}\, C(x/\lambda)$$

This equation implies that $C(x)$ is a pure power law

$$C(x) = \frac{a}{|\vec{x}|^{2A}}$$

When T is not exactly equation to T_C, the system has a finite length of correlations. When x reaches the correlation length, the correlation function departs from its power-law form and begins to decrease exponentially. The correlation length depends on the parameter

$$t = \frac{T - T_c}{T_c}$$

If the correlation length is large compared to the underlying lattice spacing, the dependence on ξ or t might also be described in the framework of scale invariance. We would expect that a scale transformation could be compensated by a rescaling of the spins and also a change in t, such that the theory at larger distances would have a smaller correlation length and thus a larger value of t. An appropriate scaling law is

$$C(x,t) = \lambda^{-2A} (x/\lambda , t\lambda^{B})$$

In a moment, I will discuss the implications of this equation for the behavior near the critical point.

Power-law dependence such as that above is seen in a wide variety of physical systems. These include systems far outside of the realm of equilibrium statistical mechanics.

1. In geography, *Mandelbrot* pointed out that the length of a coastline or the length of rivers in a river basin is gives by a nontrivial power of the distance scale. Rivers—and also arteries in the body—are 1-dimensional objects that must fill a 2-or 3-dimensional region densely.

2. In fluid mechanics, *Kolmogorov* argued that, in fully-developed turbulent motion, the velocity-velocity correlation function is scale-invariant with

$$\langle v(k)\, v(-k) \rangle \sim \frac{1}{|\vec{k}|^{5/3}}$$

3. In dynamical systems, *Feigenbaum* discovered a description of the transition to chaos in which a periodic motion goes through a succession of transitions, each of which doubles the period of the motion. The parameter distances to the next transition, and the amplitudes of the perturbations that appear at that transition, follow power laws indicating scale-invariant behavior.

4. In cosmology, *Harrison* and *Zeldovich* proposed that the primordial density fluctuations from which galaxies and other cosmic structures formed had a scale-invariant Fourier spectrum

$$\left\langle \frac{\delta\rho}{\rho}(k)\, \frac{\delta\rho}{\rho}(-k) \right\rangle \sim |\vec{k}|^{1-n} \qquad n \approx 1$$

Among these behaviors, the scale-invariant structure at a critical point is the best understood, since we can apply to this system all of the machinery of equilibrium statistical mechanics. These methods give a great deal of information about the description of scale-invariant fluctuations and suggest methods for computing the power laws. I will now present some elements of that theory.

To begin, I will return to the expression above of scale-invariance in the spin-spin correlation function. The exponents A and B that we defined there must be related to the various power laws that we found describe non-analytic behavior at the critical point. We can readily work out the connection.

I have already noted that the various power laws with which thermodynamic functions diverge at the critical point are called *critical exponents.* I will first present the standard notation for these exponents. In correlation functions, the behavior at $T = T_C$ is written

$$\langle s(x)\, s(0) \rangle \sim \frac{1}{|\vec{x}|^{d-2+\eta}}$$

The behavior for $T > T_C$ with $t = (T - T_C)/T_C \ll 1$ is

$$\langle S(x)\, S(0) \rangle \sim e^{-|\vec{x}|/\xi(T)}$$

with

$$\xi(T) \sim t^{-\nu}$$

In thermodynamic quantities, where M is the order parameter and h is the corresponding external field, we have as $T \to T_C$ from below

$$M \sim |t|^{\beta}$$

as $T \to T_C$ from above,

$$c_V \sim t^{-\alpha} \qquad\qquad \chi \sim t^{-\gamma}$$

and, as $h \to 0$ at $T = T_C$,

$$M \sim h^{1/\delta}$$

We have worked out the values of all of these exponents in Landau theory. For the correlation function exponents, we found

$$\eta = 0 \qquad \nu = \tfrac{1}{2}$$

and for the thermodynamic exponents, we found

$$\alpha = 0 \qquad \beta = \tfrac{1}{2} \qquad \gamma = 1 \qquad \delta = 3$$

We also saw that, in exactly solvable models, the critical exponents can take values different from those in Landau theory. For example, in the 2-dimensional Ising model

$$\eta = 1 \qquad \nu = 1 \qquad \alpha = 0 \qquad \beta = \tfrac{1}{8}$$

Experiment has interesting things to say about the values of the critical exponents. In three dimensions, there are measurements of critical exponents in many systems to two significant figures. In addition, it is possible to determine critical exponents in lattice models by extrapolating the the high-temperature to the singularity at $T = T_C$. The resulting exponents do not seem to be simple rational numbers. The results do fall into groups in which the exponents seem to be identical:

{ fluids (critical endpoint)
β-brass
Ising lattice models }
$\alpha = 0.11 \quad \beta = 0.33 \quad \gamma = 1.24$
$\eta = 0.03 \quad \nu = 0.63$

{ He^4
XY lattice models }
$\alpha = -0.01 \quad \nu = 0.67$

{ ferromagnets, antiferromagnets
Heisenberg lattice models }
$\alpha = -0.01 \quad \beta = 0.37 \quad \gamma = 1.40$
$\eta = 0.04 \quad \nu = 0.71$

It is striking that, although these answers disagree with the predictions of Landau theory, *models with the same Landau theory have the same values of the critical exponents.* This result is called *universality.* Models with the same values of the critical exponents are said to be in the same *universality class.*

The analysis of lattice models in 4 dimensions and above seems to give critical exponents in agreement with Landau theory.

With this introduction, I will work out the implications of scale-invariance, as expressed in our earlier formula for the correlation function, on the values of the critical exponents. The formula above implies that

$$C(x,t) = \frac{1}{x^{2A}} f(x t^{1/B})$$

where $f(z)$ is an unknown function of a single variable. If $f(z)$ has a definite limit as $t \to 0$, then the correlation function becomes a power law in that limit and we can identify

$$d - 2 + \eta = 2A$$

The function $f(z)$ cuts off this power law when $z \sim 1$. This corresponds to a distance

$$x \sim t^{-1/B}$$

Thus the correlation function exponents are given in terms of A and B by the expressions

$$\eta = 2A + 2 - d \qquad \nu = \frac{1}{B}$$

From the expression for $C(x)$, we can also derive a formula for the susceptibility exponent γ in terms of A and B. The susceptibility is given by the linear response formula

$$\chi = \int d^d x \quad \langle S(x)\, S(0) \rangle$$

Just at $T = T_C$, this becomes

$$\chi = \int d^d x \quad \frac{a}{|x|^{d-2+\eta}}$$

an integral that diverges at large x. For $t \neq 0$, the divergence is cut off by the exponential decay of correlations at $x \sim t^{-\nu}$. Then

$$\chi \sim x_0^{\,2-\eta}$$

where

$$x_0 \sim t^{-\nu}$$

This implies that

$$\chi \sim t^{-\nu(2-\eta)}$$

or

$$\gamma = \nu(2-\eta)$$

To find relations for the remaining exponents α, β, δ, we need a slightly stronger assumption about the implications of scale invariance. We need to know how the scale-invariant dynamics affects the value of the Gibbs free energy. A straightfoward assumption is that in the *Gibbs free energy density*, which has the dimensions $(\text{length})^{-d}$ in d dimensions, the length scale that provides the dimensions is the correlation length. Then, at $M = 0$,

$$G \sim \left[\frac{1}{\xi(T)}\right]^{d} \sim t^{d\nu}$$

Essentially, we assume that the leading effect in G near the critical point are the scale invariant fluctuations that determine the long-ranged part of the correlation function. When M is non-zero, G can also depend on M. Under the change of scale

$$x \to x/\lambda \qquad t \to \lambda^{1/\nu} t \qquad s \to \lambda^{\frac{d-2+\eta}{2}} s$$

so if the leading term in $G(M)$ is to be scale-invariant, it must depend on M and t in the combination

$$t^{\nu} M^{-\frac{2}{d-2+\eta}} \qquad \text{or} \qquad M\, t^{-\frac{(d-2+\eta)}{2}\nu}$$

Thus, finally, the singular part of $G(M)$ should have the form

$$G \sim t^{d\nu}\, g\left(M t^{-\frac{(d-2+\eta)\nu}{2}}\right)$$

From this, we can determine the values of the exponents α, β, and δ For α, set $M = 0$ and differentiate twice with respect to the temperature. Then

$$C_V \sim \left.\frac{\partial^2 G}{\partial t^2}\right|_{M=0} \sim t^{d\nu - 2}$$

so

$$\alpha = 2 - d\nu$$

For β, note that the value of the spontaneous magnetization below T_C will occur at the *minimum* of the function $g(z)$ for $t < 0$. This occurs at a fixed value of z and thus at a value of M that obeys

$$M \sim t^{\frac{(d-2+\eta)}{2}\nu}$$

Then

$$\beta = \frac{(d-2+\eta)\nu}{2}$$

The relation between M and h can be found from $h = \partial G/\partial M$. This implies that

$$h \sim t^{d\nu}\, t^{-\frac{(d-2+\eta)}{2}\nu}\, g'(Mt^{-\beta})$$

$$\sim t^{\frac{(d+2-\eta)}{2}\nu}\, g'(Mt^{-\beta})$$

We can write this relation equally well as

$$h \sim M^{\frac{1}{\beta}\frac{d+2-\eta}{2}\nu} \; k(tM^{-1/\beta})$$

where $k(z)$ is an unknown function. Assuming that $k(z)$ has a smooth limit as $z \to 0$, we find

$$h \sim M^{\frac{d+2-\eta}{d-2+\eta}}$$

or

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

This same equation allows us to check the relation given above for γ. Taking

$$\frac{\partial h}{\partial M} \sim t^{\frac{(d+2-\eta)}{2}\nu} \, t^{-\frac{(d-2+\eta)}{2}\nu} \; g''(Mt^{-\beta})$$

$$\sim t^{\nu(2-\eta)} \; g''(Mt^{-\beta})$$

we find

$$\gamma = \nu(2-\eta)$$

which confirms the result for γ derived earlier.

We have now found formulae for all of the critical exponents in terms of the scaling exponents A and B, or, equivalently, in terms of the correlation length exponents η and ν. All of these relations are obeyed by the exponents actually measured in 3 dimensions. The value of γ obtained in Landau theory is consistent with the values of η and ν from Landau theory.

$$\gamma\Big|_{\text{Landau}} = 1 = \frac{1}{2}(2-0) = \nu(2-\eta)\Big|_{\text{Landau}}$$

However, the relations for α, β, and δ are inconsistent with the predictions of Landau theory. In particular, the relations depend on the dimensionality, while the Landau theory predictions for the critical exponents are independent of d. Oddly, though, the relations above may be seen to be consistent with Landau theory in 4 dimensions. For example,

$$\delta\Big|_{\text{Landau}} = 3 = \left(\frac{d+2-\eta}{d-2+\eta}\right)\Big|_{d=4,\,\eta=0}$$

Above 4 dimensions, assuming that ν indeed becomes equal to $\frac{1}{2}$ in $d = 4$, the factor $t^{d\nu}$ is smaller than an analytic term proportional to t^2. Then the argument above for the values of α, β, and δ would no longer be valid.

There is a subtlety in the application of this theory to critical behavior in superfluids. There is no field that couples to the superfluid order parameter, which is essentially a Schrödinger wavefunction. Thus, there is no way to measure β or γ for a superfluid. It is possible to measure the behavior of the superfluid density as $T \to T_C^-$, through the variation of the supercurrent density. The relation we derived from Landau theory was

$$\vec{J} = |\Phi_0|^2 \frac{\hbar\vec{k}}{m} = |\Phi_0|^2 \vec{v}$$

However, taking this formula literally and replacing Φ_0 with t^β does not give the correct scaling law. Instead, we should realize that the current is absolutely normalized by the equation of number conservation

$$\frac{\partial}{\partial t} J^0 + \vec{\nabla}\cdot\vec{J} = 0 \qquad \int d^3x\, J^0 = N$$

so the order parameter rescaling exponent A or η should not enter its scaling law. The current has the form

$$\vec{J} = \rho(T)\,\vec{\nabla}\phi$$

where ϕ is the phase of the order parameter. This is derived from an effective free energy of the form

$$G = \int d^d x\ \tfrac{1}{2}\rho(T)\,(\vec{\nabla}\phi)^2$$

Then, using the same logic that we applied to $G(M)$, we should expect $\rho(T)$ to scale as $\xi^{-(d-2)}$. This gives

$$\rho(T) \sim t^{\nu(d-2)}$$

This scaling is obeyed for He^4 near its superfluid transition, where one finds

$$\rho(T) \sim t^{\zeta} \qquad \zeta = 0.67$$

Now we have computed all of the critical exponents in terms of the basic scaling exponents of the correlation function A and B, To go further, we need to develop methods to compute A and B from statistical mechanics.

A way to begin this study is to work out the behavior of the 1-dimensional Ising model under a change of scale. The 1-dimensional Ising model has no phase transition at finite temperature, but it has a correlation length that goes to infinity as $T \to 0$. Earlier in the course, we found

$$\xi(T) \sim e^{2\beta J}/2$$

Can we recover this behavior directly by the implementation of a change of scale?

In the earlier lecture, we solved the 1-dimensional Ising model using the transfer matrix method. Here, I will apply a variant of this method better suited to the discussion of scale invariance. The partition function of the model is

$$Z = \sum_{s_i=\pm 1} \exp\left[\sum_i \beta J s_i s_{i+1}\right]$$

Each exponential can be rewritten as

$$e^{\beta J s_i s_{i+1}} = \cosh \beta J + s_i s_{i+1} \sinh \beta J$$

$$= (\cosh \beta J)(1 + s_i s_{i+1} z)$$

where

$$z = \tanh \beta J$$

Then

$$Z = (\cosh \beta J)^N 2^N \prod_i \frac{1}{2} \sum_{s_i=\pm 1} \prod_i (1 + s_i s_{i+1} z)$$

Now we can manually change the distance scale in the model by summing over every other spin.

This converts the original model into a model in which the underlying lattice spacing is twice as large. A typical spin sum is

$$\frac{1}{2}\sum_{s_i}\left(1+z s_{i-1}s_i\right)\left(1+z s_i s_{i+1}\right)$$

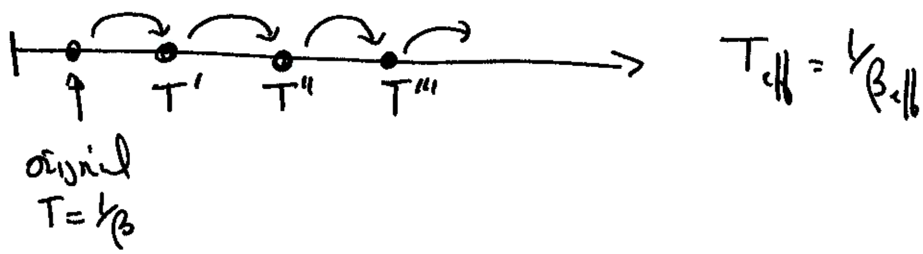
$$=\frac{1}{2}\left[\left(1+z s_{i-1}\right)\left(1+z s_{i+1}\right)+\left(1-z s_{i-1}\right)\left(1-z s_{i+1}\right)\right]$$

$$=1+z^2 s_{i-1}s_{i+1}$$

The result of these sums is a 1-dimensional Ising model of the original form but with

$$z' = z^2 \qquad \text{or} \qquad \tanh(\beta J)' = \tanh^2 \beta J$$

Notice that the new model is at higher temperature. This transformation defines a flow from large β to smaller values of β.



If we start with a model that is very close to zero temperature, then

$$\tanh \beta J \approx 1 - 2e^{-2\beta J}$$

After n steps, we reach a model with temperature given by

$$\tanh(\beta J)^{(n)} = [\cdot[(1-2\,e^{-2\beta J}+\dots)^2]^2\cdot\cdot]^2 \overbrace{}^{n \text{ times}} = 1 - 2^{n+1}e^{-2\beta J} + \dots$$

When his temperature becomes of order 1 in units of J^{-1}, the correlation length will be of order 1 in current lattice spacings. In terms of the original lattice spacing, this gives a correlation length of

$$\xi \cong 2^n$$

The value of n required to reach this situation is given by

$$2^{n+1} \cong e^{2\beta J}$$

Thus,

$$\xi = \frac{e^{2\beta J}}{2}$$

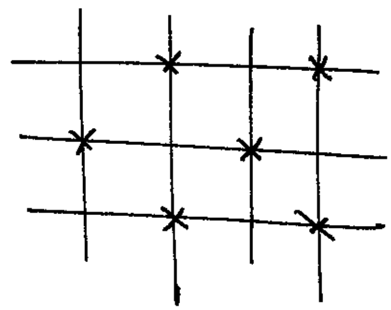
in agreement with our earlier exact solution.

This method of analyzing statistical mechanical models by summing over degrees of freedom to create a new model with a larger basic length scale is called — for historical reasons too complex to explain – a *renormalization group transformation.* A scale-invariant system is one that looks that same at every distance scale. Such a system will be left *invariant* by the renormalization group transformation. This turns out to be a powerful method for finding and analyzing scale-invariant models.

To further illustrate the renormalization group (henceforth, RG), I will now apply the same method to the calculation of the partition function for the 2-dimensional

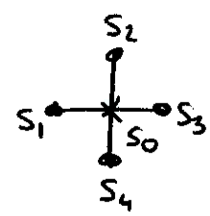
Ising model. Since the final result must be a function of the combination (βJ), it will be useful to simplify the notation by setting $J = 1$.

We can thin a 2-dimensional lattice by a factor of 2 by summing over alternate spins in a pattern such as



The calculation is more complicated than in the 1-dimensional case, however. A typical spin sum is now

$$\frac{1}{2}\sum_{s_0} (1 + s_0 s_1 z)(1 + s_0 s_2 z)(1 + s_0 s_3 z)(1 + s_0 s_4 z)$$



$$= 1 + z^2 (s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4) + z^4 s_1 s_2 s_3 s_4$$

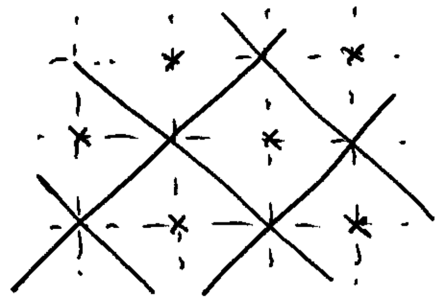
The result is

$$= C' \exp\left[\beta' (s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4) + \gamma' (s_1 s_2 s_3 s_4) + \cdots \right]$$

where

$$\beta' = \tanh^{-1} z^2 = \tanh^{-1}(\tanh^2 \beta)$$

and C', γ' are more complicated to compute. The first term in the exponent gives a nearest-neighbor interaction on the new lattice



$$a' = \sqrt{2}\,a$$

Each new bond gets a contribution from from each of two spin sums, so the strength of each new interaction is

$$\beta' = 2 \tanh^{-1}(\tanh^2 \beta)$$

There is also a 4-spin interaction that was not present previously. This will feed back and affect the 2-spin interaction at the next stage of the transformation. Further iterations of the RG transformation will also generate 6-spin, 8-spin, and higher interactions that must also be accounted for.

It is interesting, though, to ignore the 4- and higher-spin terms and study the consequences of the simple recursion formula for the coefficient of the 2-spin interaction.

$$\beta' = 2 \tanh^{-1}(\tanh^2 \beta)$$

For small β or high T, this becomes

$$\beta' = 2\beta^2$$

and so the effective β tends to smaller values (higher temperature) at larger scales. For β large, on the other hand, we should expand

$$\tanh \beta = 1 - 2 e^{-2\beta}$$

so that the recursion formula becomes

$$\beta' = 2\tanh^{-1}(1-4e^{-2\beta})$$

$$= 2\tanh^{-1}(1-2\,e^{-2(\beta-\frac{1}{2}\log 2)})$$

$$= 2(\beta-\tfrac{1}{2}\log 2)$$

Then as we go to larger scales, β becomes larger and we go to lower temperature. For an intermediate value of β, there must be a *fixed point* where

$$\beta' = \beta = \beta_*$$

Substituting values, we find

$$\beta_* = 0.61$$

At this point, the effective Hamiltonian describing the interactions of spins at any scale would be identical. That is, the fixed-point would describe a scale-invariant situation.

If β tends to smaller values at large scales, the system becomes more and more disordered. In this case, the large-scale behavior is a disordered system with $\langle S_i \rangle = 0$. If β tends to larger values at large scales, the system becomes more ordered. For $\beta \to \infty$, the system has $\langle S_i \rangle = \pm 1$, so the whole region in which the recursion formula sends β to higher values has $|\langle S_i \rangle| > 0$. The fixed point is the boundary between these behaviors. Thus, the fixed point should be identified with the critical temperature T_C or β_C. Earlier in the course, we found that, for the 2-dimensional Ising model

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}) = 0.44$$

Our crude calculation thus does not do so badly at estimating β_c. (Mean field theory, as you will recall, gives $\beta_c = 1/2d = 0.25$.)

Further analysis of the recursion formula allows us to make an estimate of the critical exponent ν. To do this, we should compute the stability behavior of the fixed point. A value of β close to the fixed point β_* will produce a value of β' that is also close to β_*. The linear term in the relation is

$$\Delta(\tanh \beta'/2) = \Delta(\tanh^2 \beta)$$

$$\frac{1}{2}\frac{1}{\cosh^2 \beta'/2}\,\Delta\beta' = 2\frac{1}{\cosh^2\beta}\tanh\beta\,\Delta\beta$$

$$\Delta\beta = \beta - \beta_*$$

$$\Delta\beta' = \left(\frac{4\tanh\beta\cosh^2\beta'/2}{\cosh^2\beta}\right)\Delta\beta$$

Evaluating the coefficient at β_*, we find

$$\Delta\beta' \cong (1.68)\,\Delta\beta$$

Notice that the fixed point is unstable in β, consistent with the analysis above. After n iterations, a small deviation $\Delta\beta$ has grown to

$$(\Delta\beta)^n = (1.68)^n\,\Delta\beta$$

The lattice spacing at this stage is related to the original lattice spacing by

$$a^{(n)} = 2^{n/2}\,a$$

When $\Delta\beta$ becomes of order 1, the correlation length will be of order 1 in units of the spacing of the current lattice. Then

$$\xi = a\, 2^{n/2} \quad \text{when} \quad (1.68)^n(\beta-\beta_*) \sim 1$$

This implies

$$\xi = a\, e^{n \cdot \frac{1}{2}\log 2} = a\, e^{\left(\frac{\log 2}{2 \log 1.68}\right)\left[-\log \Delta\beta\right]}$$

so that

$$\xi \sim (\Delta\beta)^{-0.67}$$

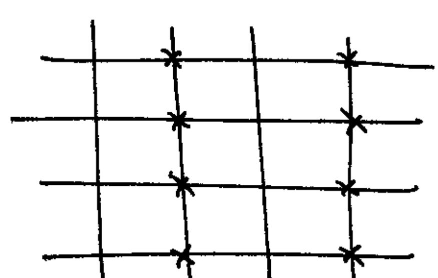
The original small difference $\Delta\beta$ is just proportional to $t = (T - T_C)/T_C$. Thus, we find

$$\xi \sim t^{-\nu} \qquad \nu \cong 0.67$$

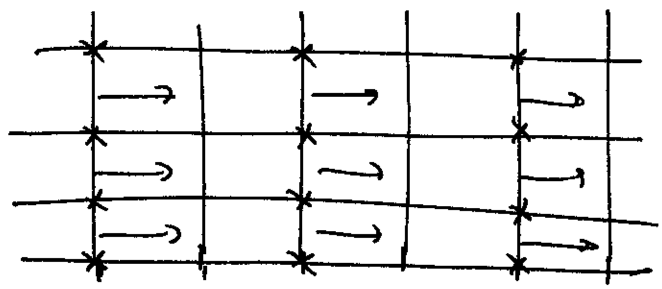
To be compared to $\nu = 1$ in the exact solution to the 2-dimensional Ising model.

Migdal proposed a less crude approximation scheme for the RG transformation that is still analytically tractable (JETP 42, 413 and 743 (1976)). It leads to an interesting analysis that I will now describe.

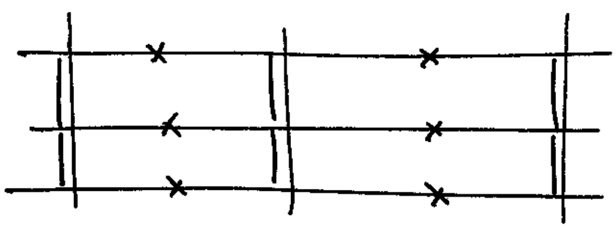
Migdal chose to thin the lattice by a factor of 2 by removing columns of spins



If we were to do this exactly, the sum over the horizontal bonds would be easy and would give just the result that we found in the 1-dimensional Ising model. However, the vertical bonds, which couple neighboring rows, complicate the calculation. Migdal suggested the approximation of moving these out of the way and doubling the strength of the interactions that are not summed over at this stage.



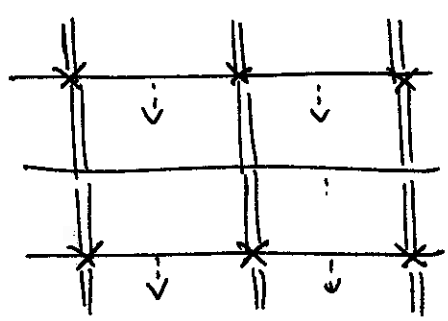
giving



Now we can easily sum over the marked spins and find

$$Z' = Z^2$$

as in the 1-dimensional case. Next, we need to thin the lattice in the vertical direction by summing over rows of spins.



To do this, move the awkward horizontal bonds out of the way. This then doubles the strength of the horizontal bonds. Before the move the value of β for the horizontal bonds was

$$\beta_{H} = \tanh^{-1}(\tanh^2 \beta)$$

so the full recursion formula for β is

$$\beta' = 2 \tanh^{-1}(\tanh^2 \beta)$$

This is just the transformation we derived earlier, except that the change of scale is a factor of 2 rather than $\sqrt{2}$. So, we again find

$$\beta_* = 0.61$$

and, because of the new scaling

$$\nu = 1.34$$

The Migdal approximation has the advatage that it is much easier to generalize to other schemes of integrating out. It can also be used as the first term in a systematic expansion for the exact RG transformation.

For the Ising case, we could imagine modifying the Migdal transformation so that, instead of changing the lattice scale by a factor 2, we changed the scale by a factor λ. We can immediately write the Migdal approximate recursion formula for this choice,

$$\beta' = \lambda \tanh^{-1}(\tanh^{\lambda} \beta)$$

This formula is derived for integers $\lambda = 2, 3, \cdots$, but I will now apply it for a continuous change of scale

$$\lambda = 1 + \epsilon$$

Expanding for small ϵ,

$$\beta' = (1+\epsilon)\tanh^{-1}\left[\tanh\beta\,(1+\epsilon\log\tanh\beta)\right]$$

$$\beta' - \beta = \epsilon\tanh^{-1}(\tanh\beta) + \frac{1}{1-\tanh^2\beta}\,\epsilon\tanh\beta\log\tanh\beta$$

$$= \epsilon\left[\beta + \sinh\beta\cosh\beta\,\log\tanh\beta\right]$$

Then

$$\beta' - \beta = \epsilon\left[\beta + \tfrac{1}{2}\sinh 2\beta\cdot\log\tanh\beta\right]$$

and so we find a differential equation for the effective value of β

$$\lambda\frac{d}{d\lambda}\beta(\lambda) = \beta + \tfrac{1}{2}\sinh 2\beta\cdot\log\tanh\beta$$

The fixed point occurs where the right-hand side of the equation vanishes. Oddly, if we put in the *exact* value of the transition temperature in the 2-dimensional Ising model

$$\beta_c = \tfrac{1}{2}\log(1+\sqrt{2})$$

we have

$$\sinh 2\beta_c = 1 \qquad \cosh 2\beta_c = \sqrt{2} \qquad \log \tanh \beta_c = -\log(1+\sqrt{2})$$

$$\frac{1}{2}\sinh 2\beta_c \log \tanh \beta_c = -\frac{1}{2}\log(1+\sqrt{2}) = -\beta_c$$

and so this is precisely the location of the fixed point!

Again, we can compute the exponent ν by linearizing the recursion formula about the fixed point. This gives

$$\lambda \frac{d}{d\lambda}(\beta-\beta_*) = (\beta-\beta_*) + \frac{1}{2}\sinh 2\beta_* \frac{1}{\tanh \beta_*} \frac{\beta-\beta_*}{\cosh^2 \beta_*}$$
$$+ \cosh 2\beta_* (\beta-\beta_*) \log \tanh \beta_*$$

or

$$\lambda \frac{d}{d\lambda} \Delta\beta = \left[1 + 1 - \sqrt{2}\log(1+\sqrt{2})\right] \Delta\beta$$

so that finally

$$\lambda \frac{d}{d\lambda} \Delta\beta = (0.754)\, \Delta\beta$$

The solution of this equation is

$$\Delta\beta(\lambda) = \lambda^{0.754} (\Delta\beta)(\lambda=1)$$

The correlation length is the length scale λa for which $\Delta\beta(\lambda)$ is of the order of 1. Then

$$\xi \sim [\Delta\beta(1)]^{-1/0.754}$$

or

$$\xi \sim t^{-\nu} \qquad \nu = 1.33$$

This value, again, should be compared with $\nu = 1$ in the exact solution.