May 6

Landau Theory with Continuous Symmetry

In the previous lecture, I described Landau's phenomenological theory of the critical point and order-disorder transitions. I noted that the order parameter of Landau theory can have several components that transform into one another under a continuous symmetry group. This brings in new physical effects that I will now describe.

At the end of the last lecture, we computed the form of a *domain wall* giving the boundary between the two ordered phases of the Ising model. I will now consider the generalization of this problem to the XY model. We can begin from the analogous starting point

$$G = \int d^3x \; \frac{1}{2}\rho(\vec{\nabla} M^i)^2 + \frac{1}{2}a(T-T_c)(M^i)^2 + \frac{1}{4}b((M^i)^2)^2$$

where $i = 1,2$ for the XY case. This effective free energy is invariant under 2-dimensional rotations that mix M^1 and M^2. The variational equation is

$$-\rho\nabla^2 M^i + a(T-T_c)M^i + bM^i(M^j)^2 = 0$$

We need to find the solution $M^i(z)$ with the boundary conditions

$$M^i(z=-L) = (-M_0, 0) \qquad M^i(z=L) = (+M_0, 0)$$

where

$$M_0 = \left[\frac{a(T_c-T)}{b}\right]^{1/2}$$

and L is very large, that has minimal free energy. One solution of the differential equation is the generalization of the solution that we found in the previous lecture

$$M^i(z) = \left(M_0 \tanh \frac{z}{2\xi(T)} , 0\right)$$

However, there is a better solution

$$M^i(z) = M_0 \left(\sin \frac{\pi}{2}\frac{z}{L} , \cos \frac{\pi}{2}\frac{z}{L}\right)$$

This is a slow rotation of M^i from on end of the sample to the other. Since $|\vec{M}| = M_0$ everywhere, this costs *zero* potential energy in G. Also

$$\frac{1}{2} \rho(\vec{\nabla} M^i)^2 = \frac{1}{2}\left(\frac{\pi}{2L}\right)^2 M_0^2$$

so the total cost of free energy is

$$\Delta G = \rho \frac{\pi^2}{8L^2} M_0^2 \cdot 2L \cdot \text{Area} \sim \frac{\text{Area}}{L}$$

The cost per unit area goes to zero for a large system. The prediction of Landau theory, then, is that the domain wall evaporates into a smooth and delocalized transition.

This example illustrates two new issues, one local in the configuration space, one global. The local issues is that there are configurations near each ground state with very low values of G that are not present in the Ising case. We can take advantage of these states to smooth the transition between states with different orientations of the order parameter.

To analyze this point, consider $T < T_C$ and expand $G[M]$ about one of its minima. I will do this for the general problem of an N-component order parameter. Here

$$a(T_c - T) M^i = b M^i (M^j)^2$$

where $i = 1, \cdots, N$. Any configuration

$$M^i = \hat{n}^i M_0 \qquad M_0 = \left[\frac{a(T_c - T)}{b}\right]^{\frac{1}{2}}$$

is a minimum of G, and all of these minima are equivalent by the symmetry of the model, rotational symmetry in N dimensions. It is most convenient to analyze the particular minimum in which

$$M^i = (\underset{1}{0}, \underset{2}{0}, \cdots \underset{N-1}{0}, \underset{N}{M_0})$$

To discuss fluctuations about this state, write

$$M^i = (0\,0 \cdots 0\,M_0) + (\pi^i(x), m(x))$$

with $i = 1, \cdots, (N-1)$. Now expand $G[M]$ to quadratic order in $m(x)$ and $\pi^i(x)$. The derivative term becomes

$$\frac{1}{2}\rho\,(\vec{\nabla} M^i)^2 = \frac{1}{2}\rho\left[(\vec{\nabla}\pi^i)^2 + (\vec{\nabla} m)^2\right]$$

The quadratic term in the potential becomes

$$\frac{1}{2}a(T-T_c)(M^i)^2 = \frac{1}{2}a(T-T_c)\left\{M_0^2 + 2M_0 m + m^2 + (\pi^i)^2\right\}$$

The quartic term in the potential becomes

$$\frac{1}{4} b ((M^i)^2)^2 = \frac{1}{4} b \left[(M_0+m)^2 + (\pi^i)^2 \right]^2$$

$$= \frac{1}{4} b \left[M_0^4 + 4 M_0^3 m + 6 M_0^2 m^2 + 2 M_0^2 (\pi^i)^2 + \cdots \right]$$

The full potential is

$$\frac{1}{2} a (T-T_c)(M^i)^2 + \frac{1}{4} b ((M^i)^2)^2$$

$$= \left[-\frac{a}{2}(T_c-T) M_0^2 + \frac{b}{4} M_0^4 \right] + \left[-a(T_c-T) M_0 + b M_0^3 \right] \cdot m$$

$$+ \left[-\frac{1}{2} a (T_c-T) + \frac{3}{2} b M_0^2 \right] m^2 + \left[-\frac{a}{2}(T_c-T) + \frac{b}{2} M_0^2 \right] (\pi^i)^2 + \cdots$$

The first term is a constant; I will drop it from here on. The second term vanishes by the condition for the minimum of G,

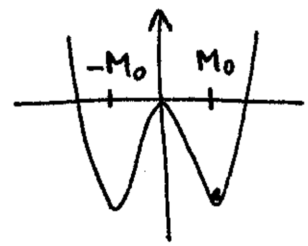
$$-a(T_c-T) + b M_0^2 = 0$$

This is required, since, if $\vec{M}$ satisfies the variational equation, the term linear in the variation must vanish. But notice that setting M_0 to the minimum value also causes the coefficient of $(\pi^i)^2$ to vanish. The final result is

$$G = \int d^3x \left\{ \frac{1}{2} \rho (\vec{\nabla} m)^2 + \frac{1}{2} \left[2a(T_c-T) \right] m^2 + \frac{1}{2} \rho (\vec{\nabla} \pi^i)^2 + (\text{cubic} + \text{higher}) \right\}$$

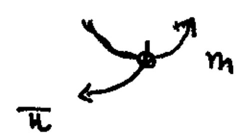
Then the fluctuations $\pi^i(x)$ do not cost any energy except through their gradients.

We can understand this structure geometrically by looking at the shape of the potential energy surface of G. In the Ising case, M has only one component, are we can draw the potential as



If we expand $M(x) = M_0 + m(x)$, the potential has positive curvature for deviations of $m(x)$ from the minimum $m(x) = 0$.

In the case of a multidimensional order parameter, the potential part of G has the form



The field $m(x)$ is the fluctuation of the order parameter in the radial direction. This is still restrained by the positive curvature of the potential. The fields $\pi^i(x)$ are the fluctuations in the directions orthogonal to the radial direction that point around the ring or sphere at the bottom of the potential well. It costs no free energy to move in these directions.

Just as we added a term to the free energy to describe slow space-dependence in the value of $\vec{M}(x)$, we can add a term to describe slow time-dependence. This would modify the variational equation

$$-\rho\nabla^2\pi^i = 0 \quad \Rightarrow \quad \left(\kappa\frac{\partial^2}{\partial t^2} - \rho\nabla^2\right)\pi^i = 0$$

This is the wave equation, with sound speed

$$C = (\rho/\kappa)^{1/2}$$

The fluctuations of π^i, then, for waves of variation of the order parameter. At low temperatures, we should quantize these waves and treat the resulting particles with quantum statistics. The particles are *bosons* and, since they can be freely created and destroyed, they have $\mu = 0$. Thus, they form a free Bose-Einstein gas like photons or phonons. They give a contribution to the specific heat C_V proportional to T^3. These particles can be the dominant carriers of heat, momentum, and other conserved quantities in an ordered medium at very low temperature.

Jeffrey Goldstone realized that the appearance of such low-energy degrees of freedom is a general property of systems with a continuous global symmetry that is spontaneously broken. Thus, we call these particles *Goldstone bosons.* In particular systems, these bosons have specific names characteristic of the type of ordering. In a ferromagnet, these particles are called *magnons* or *spin waves.* In a superfluid, they are called *phonons*—even though they are not density fluctuations.

The correlation functions of $m(x)$ fall off exponentially, in the way that we saw for the Ising model. However, the correlation functions of the π^i fall off more slowly. In the previous lecture, I proposed that the Green's function solution of the variational equation for $M(x)$ is a good model for the form of the correlation function. Then the π^i correlation function

$$\langle \pi^i(x)\, \pi^j(0) \rangle = \delta^{ij}\, D(x)$$

$$i, j = 1, \cdots, (N-1)$$

should be the solution of

$$-\rho \nabla^2 D(x) = A \cdot \delta(\vec{x})$$

In 3 dimensions, the solution is a Coulomb potential

$$D(x) = \frac{A/\rho}{4\pi} \frac{1}{|\vec{x}|}$$

We can use this formula to compute the effect of the Goldstone bosons on the full spin-spin correlation function in the XY model. It will be convenient to use the representation in which we write the 2-component vector (M^1, M^2) as a complex number $M = M^1 + iM^2$. Then we can parametrize $M(x)$ with $m(x)$ and $\pi(x)$ by the formula

$$M(x) = (M_0 + m(x))\, e^{i\pi(x)/M_0}$$

that makes explicit that π parametrizes a rotation of the magnetization. Since

$$|M(x)|^2 = (M_0 + m(x))^2$$

the length of the magnetization vector $\vec{M}(x)$ is independent of π in this parametrization. Then it is manifest that $\pi(x)$ has a *flat* potential to all orders.

Now we can compute

$$\langle M(x)\, M^*(0) \rangle \hat{=} \langle (M_0 + m(x))(M_0 + m(0)) \rangle \cdot \langle e^{i\pi(x)/M_0}\, e^{-i\pi(0)/M_0} \rangle$$

The first expectation value on the right-hand side has only short-range correlations. In the magnetized phase, it falls quickly to the value

$$M_0^2$$

The correlation function involving $\pi(x)$ has long-range correlations. To evaluate it, I will assume that $\pi(x)$ is a Gaussian random variable. This implies that, while the 2-point correlations of π are nonzero,

$$\langle \pi(x)\pi(0)\rangle = D(x)$$

the 4- and higher-point correlations vanish. Then we have a method for evaluating the second expectation value on the right-hand side above. Expand the exponentials

$$\langle e^{i\pi(x)/M_0} e^{-i\pi(0)/M_0}\rangle = \left\langle \sum_{m=0}^{\infty} \frac{i^m \left(\frac{\pi(x)}{M_0}\right)^m}{m!} \sum_{n=0}^{\infty} \frac{(-i)^n \left(\frac{\pi(0)}{M_0}\right)^n}{n!} \right\rangle$$

If we insist on pair up each $\pi(x)$ with a $\pi(0)$ to form a 2-point correlation function, the nonzero terms have $m = n$. There are $m!$ ways to do the pairing. Then this expression evaluates to

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\langle \pi(x)\pi(0)\rangle}{M_0^2}\right)^m = e^{\langle \pi(x)\pi(0)\rangle / M_0^2}$$

The full result also should include terms involving the 2-point correlation functions

$$\langle \pi(x)\pi(x)\rangle = \langle \pi(0)\pi(0)\rangle$$

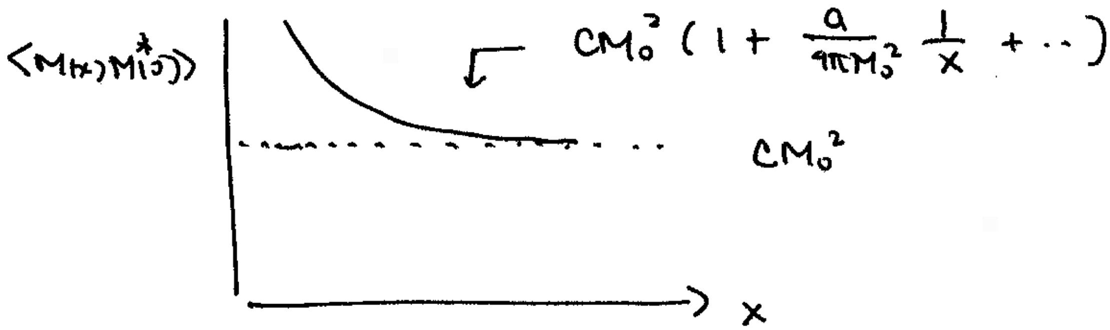
These functions do not depend on x and turn out just to contribute an overall constant. Thus, finally,

$$\langle M(x) M^*(0)\rangle \cong C\, M_0^2\, e^{\langle \pi(x)\pi(0)\rangle / M_0^2}$$

In 3 dimensions,

$$\langle M(x) M^{\dagger}(0) \rangle \cong C M_0^2 \, e^{\frac{a}{4\pi M_0^2} \frac{1}{|x|}}$$

This is a power-law decay to a nonzero asymptotic value signalling long-range order.



In a general dimensionality d, the Coulomb potential behaves as

$$D(x) \sim \frac{1}{(x)^{d-2}}$$

The limiting case of 2 dimensions is especially important. This case describes, for example, order-disorder transitions on surfaces or in layered materials. In this case, the Green's function that gives the correlation function $\langle \pi(x) \pi(0) \rangle$ is the solution of

$$- \rho \nabla^2 D(x) = A \, \delta(x)$$

which is

$$D(x) = - \frac{A}{4\pi \rho} \log\left(|\vec{x}|^2 \right)$$

Then, using the analysis above,

$$\langle M(x) M^*(0)\rangle \cong C M_0^2 \exp\left[-\frac{A}{4\pi\rho M_0^2}\log x^2\right]$$

$$= C M_0^2 \, \frac{1}{|\vec{x}|^{A/2\pi\rho M_0^2}}$$

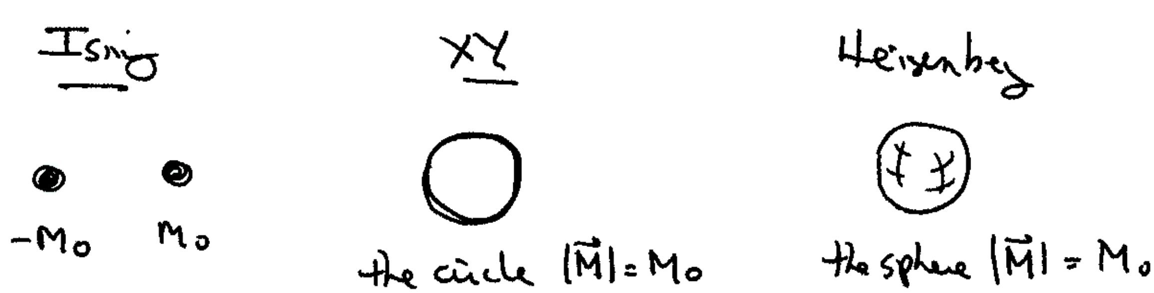
This expression tends to *zero* as $|\vec{x}| \to \infty$. There is no long-range order. In 2 dimensions, then, the fluctuations of the Goldstone boson degrees of freedom destroy long-range order for any system with a *continuous symmetry* that might be spontaneously broken. The explicit argument that I have given here is for the case of the XY model. However, if the order parameter has more components, there are more fluctuations, and therefore there is even more disorder. *Mermin and Wagner* proved this rigorously:

In 2 dimensions, a system with continous symmetry cannot have long-range order.

For some time, people believed that this theorem forbid phase transitions in the 2 dimensional XY model and in related systems such as thin films of He^4. But that is not correct, as I will discuss in a moment.

Up to now, we have discussed aspects of the behavoir of systems with continuous symmetry that are local in the configuration space. However, there are additional phenomena seen in systems with spontaneous symmetry breaking that are related to the *global topology* of the space of minima of $G[M]$.

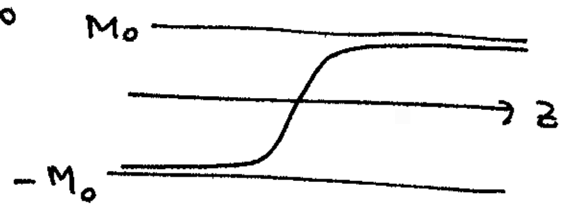
The topology of the space of minima of G is different for different the different cases of symmetry group and order parameter. For the three classes of magnets, the spaces of minima are



A smooth $\vec{x}$-dependent configuration of the order parameter $\vec{M}(\vec{x})$ can have finite free energy only if $\vec{M}(\vec{x})$ tends to a minimum of G as $|\vec{x}| \to \infty$. Thus, these spaces define boundary conditions for solutions to the variational equation for $\vec{M}(\vec{x})$. For example, for $T < T_C$, a minimum free energy configuration is $\vec{M}(x) = M_0\hat{n}$. A small ripple in $\vec{M}(\vec{x})$ has finite energy if the configuration tends to $M_0\hat{n}$ at infinity. However, there are other possible configurations that involve large deviations of $\vec{M}(\vec{x})$ from a uniform state.

We saw the simplest example of this at the end of the previous lecture. In the Ising model, the set of minima of G has two disjoint components $M = +M_0$ and $M = -M_0$. We found a *domain wall* solution of the variational equation that is the minimum free energy configuration with the boundary conditions

$$M(z) \to \begin{cases} M_0 & z \to \infty \\ -M_0 & z \to -\infty \end{cases}$$



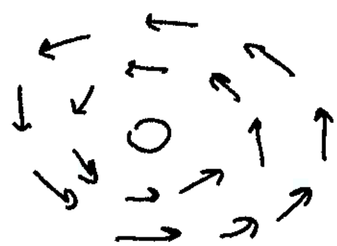
Even though this solution has higher free energy than the uniform state, the configuration is stable with respect to small deformations, since small deformations cannot change the nontrivial boundary conditions. And, those boundary conditions cannot be deformed into those of a uniform configuration; they are *topologically distinct.*

More generally, we can characterize possible boundary conditions on $\vec{M}$ by dividing them into topological classes. A nontrivial solution to the variational equation for $\vec{M}(\vec{x})$ that is the lowest free energy configuration with boundary conditions in one of these topological classes is said to be *topologically stable.*

In the XY model, the space of minima of G is a circle and is simply connected. This means that we can smoothly deform the boundary condition of the domain wall into a trivial boundary condition while keeping the boundary condition in the space of minima of G.

This is another argument, complementary to the one given at the beginning of this lecture, that sharp domain walls are not stable in the XY model.

However, in the XY model, there is a new topologically stable configuration, the *vortex.* We can imagine a cylindrically symmetric configuration in which the boundary condition on $\vec{M}$ is a function of the azimuthal angle ϕ that winds once around the circle as $\vec{x}$ winds around the circle at infinity.



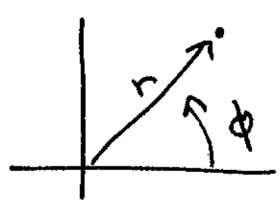
I will now find this solution from the variational equation. It will be easiest to proceed if we represent $\vec{M}$ by a complex number. Then the variational equation is

$$\nabla^2 M = -\frac{a(T_c - T)}{\rho} M + \frac{b}{\rho} M \, |M|^2$$

We can look for a solution of the form

$$M = M_0 \, f(r) \, e^{i\phi} \qquad\qquad M_0 = \left[\frac{a(T_c - T)}{b}\right]^{\frac{1}{2}}$$

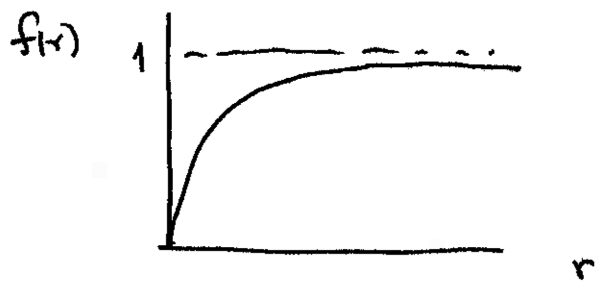
where ϕ is the azimuthal angle in space,



The equation for $f(r)$ is

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{1}{r^2}\right) f(r) = \frac{1}{2\xi^2(T)}\left[-f(r) + f^3(r)\right]$$

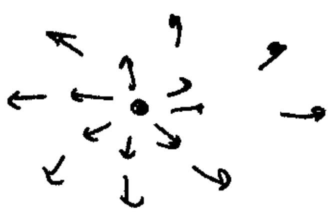
This equation has a solution of the form

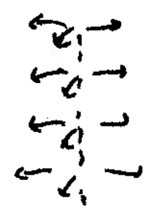


with

$$f(r) \to 0 \quad \text{as} \quad r \to 0 \qquad\qquad f(r) \to 1 \quad \text{as} \quad r \to \infty$$

It gives a vortex-like configuration in which the order parameter vanishes at $r = 0$





There are similar solutions with

$$M(\vec{x}) \sim M_0\, e^{im\phi} \qquad\qquad m = -\infty, \ldots, \infty$$

corresponding to boundary conditions that wind m times around the circle of minima of G as $\vec{x}$ goes around the circle at infinity. The case $m = -1$ is called the *antivortex.*

This configuration does not quite have finite free energy. The gradient of M is

$$\vec{\nabla} M \sim \hat{\phi}\, \frac{1}{r} \frac{\partial}{\partial \phi} \left(M_0 f(r) e^{i\phi} \right) \sim \frac{i\hat{\phi}}{r}\, M_0 f(r)$$

Then the kinetic term in G contributes

$$G \sim \int d^3x \; \tfrac{1}{2}\rho |\vec{\nabla} M|^2$$

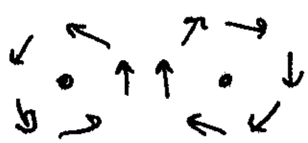
$$\sim \int dr \; r \; 2\pi \, dz \quad \tfrac{1}{2}\rho |\vec{\nabla} M|^2$$

$$\sim (\text{Length}) \cdot \pi \rho M_0^2 \int^{\infty} \frac{dr \; r}{r^2}$$

The free energy is logarithmically divergent at large distances. However, vortices can still be relevant in a sample of finite size, epscially if the boundary conditions favor the formation of vortices. We will see an example in the next lecture.

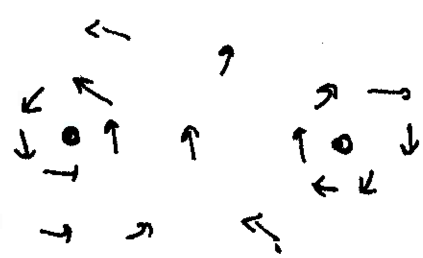
In 2 dimensions, the vortex is a particle-like excitation

The free energy of an isolated particle in a box of size L diverges like $\log L$, but there are finite-energy configurations consisting of vortex-antivortex pairs,

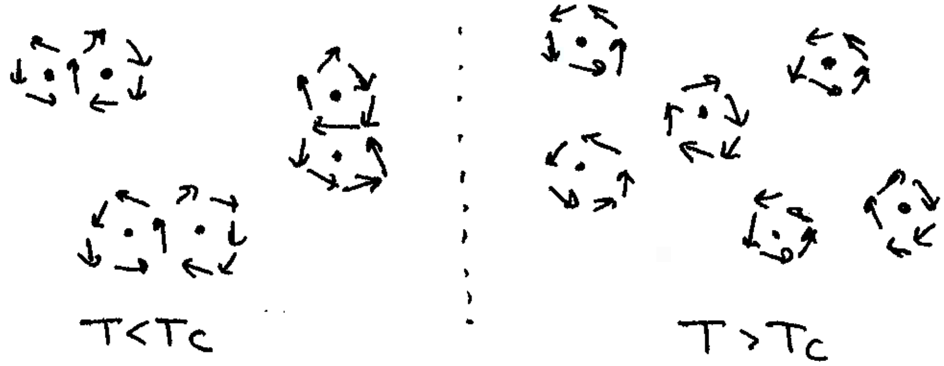


Vortices and antivortices attract each other by a potential

$$V(r) \sim \log r$$



Kosterlitz and Thouless realized that the density of vortex pairs in equilibrium grows as T increases, and the attractive potential becomes weaker. At a certain point, it is advantageous for the pairs to dissociate and form a plasma of vortices



This is the *Kosterlitz-Thouless transition.* For $T < T_C$, we have a state of the XY model that is as ordered as possible in 2 dimensions, with power-law spin correlations

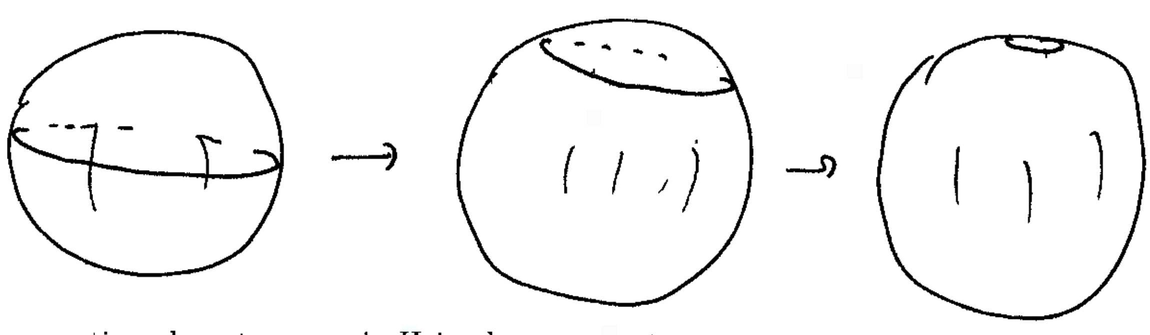
$$\langle M^i(x)\, M^j(0)\rangle \sim \delta^{ij} \frac{1}{|\vec{x}|^{a/2\pi}}$$

For $T > T_C$, the spin configuration is more randomized, giving

$$\langle M^i(x)\, M^j(0)\rangle \sim \delta^{ij} \exp\left[-|\vec{x}|/\lambda\right]$$

where λ is the screening length in the plasma. This mechanism describes the phase transition observed in He^4 films and in the liquid-solid (melting) transition in 2 dimensions.

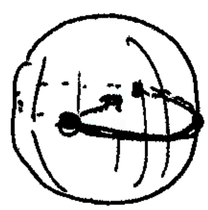
Next, consider the Heisenberg model. In this case, the vortex solution becomes unstable, since we can smoothly deform the nontrivial boundary condition of that solution to a trivial configuration while remaining at all times within the space of minima of G.



Thus, vortices do not appear in Heisenberg magnets.

However, there are vortices in the related system of the *nematic liquid crystal*. This system also has a 3-component order parameter, the vector $\vec{n}$ that gives the orientation of molecules in the liquid. However, $\vec{n}$ and $-\vec{n}$ are identical states

This means that a boundary condition in which the the order parameter winds through an angle of π at infinity gives a smooth path in the space of minima of G. More precisely, the space of minima of G is the sphere with *opposite points identified.* This sphere contains paths such as

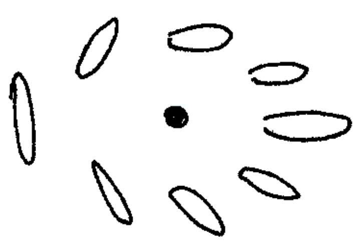


that are topologically distinct and cannot be deformed into the trivial configuration.

We can make use of one of these paths to find a topologically stable vortex solution

$$\vec{n}(x) = n_0 f(r) \left(\cos \phi/2 \quad \sin \phi/2 \, , \, 0 \right)$$

The whole configuration has a smooth join at $\phi = \pi$[1],



[1]Note that Fig. 9.14 in Sethna's book does not quite describe this solution correctly.

There are other systems with order-disorder transitions, such as *smectic* liquid crystals and superfluid He^3, that have even more complex order parameters and exhibit more exotic topologically stable solutions. This subject remains an active area of research in condensed matter physics.