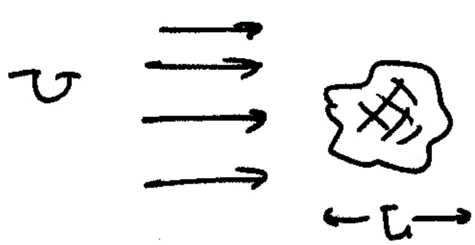
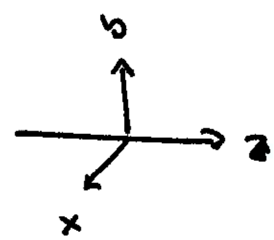
Feb. 9

# Wakes and Separation

In our discussion of boundary layers, we saw that the effect of viscosity in high Reynolds number flows is often restricted to small regions with large gradients in the velocity. Turbulent flow can also be restricted to specific regions. We have previously discussed the case of a boundary layer with laminar flow. This discussion needs to be generalized to the situation in which the flow in the narrow boundary layer crosses an instability threshold and becomes turbulent. In addition, we need to discuss the wake generated by a body moving through a high Reynolds number flow. These two topics will be the subject of this lecture.

I will begin with a discussion of the wake in the case of high Reynolds number but laminar flow. Consider a stationary body of some arbitrary shape, placed in a flow with $\vec{v} = U\hat{z}$ as $z \to -\infty$. I assume steady flow.

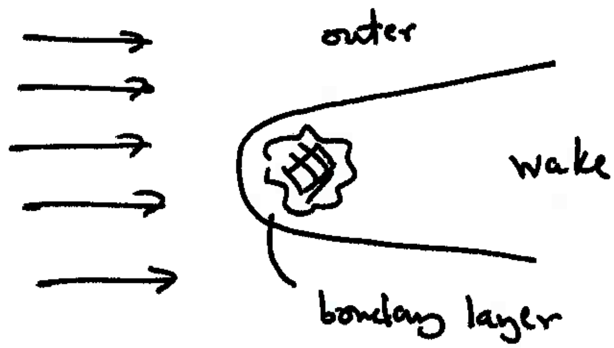



The flow has a high Reynolds number

$$\frac{UL}{\nu} \gg 1$$

In this situation, we have three regions in the flow, the outer region, the boundary layer, and the wake.



In the outer region, $\nu$ is unimportant. Also, since

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{V} + O(\nu)$$

and the vorticity is zero in the incoming flow to the far left, the vorticity remains close to zero along streamlines that move through this region to right. Thus, we have irrotational incompressible flow in this region. The velocity in this whole region is close to $\vec{v} = U\hat{z}$.

Viscosity is important in the boundary layer, and so vorticity develops there. This vorticity is carried by the flow into the wake region. Thus, even when the velocities in the wake come back to relatively small deviations from $\vec{v} = U\hat{z}$, these velocities are not described by a potential flow. This will be important at several points in this discussion.

Consideration of the wake allows us to compute the overall drag and life forces that the fluid exerts on the body. Write the fluid velocity in the form

$$\vec{V} = \vec{U} + \vec{v}$$

where $\vec{U} = U\hat{z}$ and $\vec{v}$ is the *deviation* from uniform flow. In the region of potential flow, the pressure is given by Bernoulli's formula. For steady flow

$$\frac{1}{2}(\vec{U}+\vec{v})^2 + \frac{p}{\rho} = (\text{const}) = \frac{1}{2}U^2 + \frac{p_\infty}{\rho}$$

where the last equality is the evaluation of the constant at $z \to -\infty$. Let

$$\Delta p = p - p_\infty$$

Then

$$\Delta p = -\varrho\, \vec{U} \cdot \vec{v}$$

up to terms of order $v^2$. I will apply this decomposition of the $\vec{V}$ only far away from the body where order $v^2$ deviations can be ignored. Taking $\partial/\partial z$ of this equation.

$$\vec{U} \frac{\partial}{\partial z} \vec{v} = -\frac{1}{\varrho} \frac{\partial p}{\partial z}$$

The full Navier-Stokes equation linearized about $\vec{V}$ has one more term from the viscosity

$$\vec{U} \cdot \frac{\partial}{\partial z} \vec{v} = -\frac{1}{\varrho} \frac{\partial p}{\partial z} + \nu \nabla^2 v_z$$

We will see that the viscosity term is important not only in the boudary layer but also in the wake. Thus, the importance of the pressure deviation in determining the flow is minimized. I will show later that $\Delta p$ is subdominant in the wake.

The overall momentum flow in this system can be analyzed using the stress tensor. The momentum conservation equation is

$$\frac{\partial}{\partial t} (\varrho v^i) + \nabla_k T^{ik} = 0$$

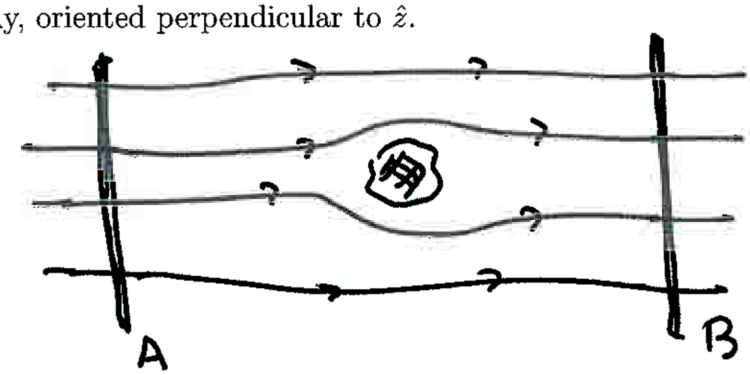
where, ignoring viscosity,

$$T^{ik} = p \delta^{ik} + \varrho V^i V^k$$

The quantity

$$\int_S d\hat{s}\; n^k T^{ik}$$

is the momentum per second flowing through the area $S$ in the direction of *hatn*. Construct surfaces $A$ to the left of the fixed body and $B$ to the right of the fixed body, oriented perpendicular to $\hat{z}$.




The net flow of momentum into the region of the body is then

$$\int_A dx\,dy\; \hat{z}^k T^{ik} \;-\; \int_B dx\,dy\; \hat{z}^k T^{ik}$$

This flow of momentum is exactly the force that the fluid exerts on the body. Expanding the expression for the stress tensor,

$$T^{ik} = (p_\infty \delta^{ik} + \rho U^i U^k) \;+\; \Delta p\, \delta^{ik}$$
$$\qquad + \rho (U^i v^k + v^i U^k) \;+\; \rho v^i v^k$$

The first tem makes no contribuiton to the force, since

$$\int_A dx\,dy\; \hat{z}^k \;-\; \int_B dx\,dy\; \hat{z}^k \;=\; 0$$

Similarly, since $\vec{\nabla} \cdot \vec{v} = 0$,

$$\int_{A-B} dx\,dy\; \hat{z}^k\, v^k \;=\; \int_{\text{interior volume}} d^3x\; \vec{\nabla}\cdot\vec{v} \;=\; 0$$

so the term with $v^k$ makes no contribution. We can choose $A$ and $B$ sufficiently far away from the body that the terms of order $v^2$ are negligible. Then the force is given by

$$F^i = \int_{A-B} dx\,dy\; \hat{z}^k \left( \Delta p\, \delta^{ik} + \rho\, U^k v^i \right)$$

Consider first $F^z$, the *drag* force,

$$F^z = \int_{A-B} dx\,dy \left( \Delta p + \rho\, U v^z \right)$$

By the equation for $\delta p$ from Bernoulli's theorem, the integrand vanishes in the region of potential flow. In particular, we have potential flow on the whole surface $A$, and on the surface $B$ outside the wake region. Then

$$F^z = -\int_B dx\,dy \left( \Delta p + \rho\, U v^z \right)$$

If, as I claimed above, the pressure term can be neglected relative to the other two terms of the Navier-Stokes equation in the wake region, then this reduces to

$$F^z = -\rho U \int_{B \cap \text{wake}} dx\, dy \; v^z$$

The sign is correct. The body slows the fluid down, so $v^z < 0$. Then the drag force is positive $F^z > 0$.

There is a similar formula for the *lift*, the force perpendicular to $\hat{z}$. In the case, the pressure term is simply absent.

$$F^y = \int_{A-B} dx\, dy \; \rho U v^y$$

Where $\vec{V}$ is described by potential flow,

$$\vec{v} = \vec{\nabla}\phi \quad , \qquad v^y = \frac{\partial}{\partial y}\phi$$

where $\phi \to 0$ at infinity. The surface $A$ is entirely in the outer region, so

$$\int_A dy\, v^y = \int_A dy\, \frac{\partial \phi}{\partial y} = \phi(y=\infty) - \phi(y=-\infty) = 0$$

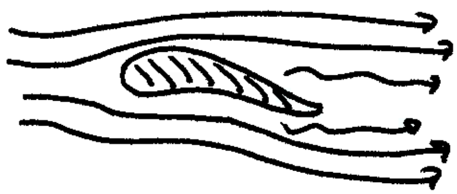
Then

$$F^y = -\rho U \int_B dx\, dy \; v^y$$

and similarly for $F^x$. If the formulae of potential flow applied also on the surface $B$, the lift would be zero. Thus, the lift can be nonzero only by virtue of the breakdown of potential flow in the wake. For a symmetric body,

$$v^y(y) = -v^y(-y)$$

behind the body. This implies $F^y = 0$. A body gives lift if it deflects the flow in the opposite direction. For example,



I will say more about lift in a moment, but first I would like to describe the flow in the wake more quantitatively. The wake is precisely the region behind the body where the vorticity is nonzero. In steady flow, vorticity obeys

$$\vec{V}\cdot\vec{\nabla}\vec{\omega} = \vec{\omega}\cdot\vec{\nabla}\vec{V} + \nu\nabla^2\vec{\omega}$$

For enough out in the wake, we can linearize about $\vec{V} = \vec{U}$. Vorticity is first order in the deviation $\vec{v}$, so

$$U\frac{\partial}{\partial z}\vec{\omega} = \nu\nabla^2\vec{\omega}$$

To solve this equation, let $\vec{v} = \vec{v}_1 + \vec{v}_2$, where $\vec{v}_1$ solves

$$U \frac{\partial}{\partial z} \vec{v}_1 = \nu \nabla^2 \vec{v}_1$$

and $\vec{v}_2 = \vec{\nabla}\Phi$, so that $\vec{v}_2$ does not contribute to $\vec{\omega} = \vec{\nabla} \times \vec{v}$. The equation for $\vec{v}_1$ is the Navier-Stokes equation ignoring pressure. Once we have found $\vec{v}_1$, I will add back $\vec{v}_2$ and show that it is consistently parametrically smaller.

Since the wake is narrow in $(x, y)$ and long in $z$, we can approximate the $\vec{v}_1$ equation as

$$U \frac{\partial}{\partial z} \vec{v}_1 = \nu \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \vec{v}_1$$

This is the diffusion equation, with the time variable replaced by $z/U$, that is, time in the zeroth-order flow. Vorticity diffuses outward in the wake. Far away from the body, the solution of this equation will be

$$\vec{v}_1 = \frac{\vec{C}}{[4\pi\nu z/U]^{2/2}} e^{-(x^2+y^2)/(4\nu z/U)}$$

This solution will be valid when the diffusion length $4\nu z/U$ is much larger than the size of the body. The 2-dimensional integral of $\vec{v}_1$ is $\vec{C}$. If $|\vec{v}_1| \gg |\vec{v}_2|$, we can identify this with the force on the body through the relations above

$$\vec{F} = -\rho U \vec{C}$$

Then

$$\vec{v}_1 = -\frac{\vec{F}}{4\pi\nu\rho z} e^{-U(x^2+y^2)/4\nu z}$$

Now that we have computed $\vec{v}_1$, we need to construct $\vec{v}_2$ and show that it is relatively small. Notice that, in the above, we have not yet imposed $\vec{\nabla}\cdot\vec{v}=0$. It is this constraint that allows us to solve for the pressure. For a drag force in the $\hat{z}$ direction,

$$\vec{\nabla}\cdot\vec{v}_1 = \frac{\partial}{\partial z}\left(-\frac{F^z}{4\pi\nu\rho z} e^{-U(x^2+y^2)/4\nu z}\right)$$

$$\sim \frac{F^z}{4\pi\nu\rho z^2} e^{-U(x^2+y^2)4\nu z} \sim O\left(\frac{1}{z^2}\right)$$

The $\vec{v}_2$ contribution must repair this. Since $\vec{\nabla}\cdot(\vec{v}_1+\vec{v}_2)=0$, we must have

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi = -\vec{\nabla}\cdot\vec{v}_1$$

Then

$$\Phi \sim \frac{1}{z} \qquad\qquad v_2^z \sim \frac{1}{z^2}$$

For large $z$, then

$$v_1^z \sim \frac{F^z}{\rho\nu z} \qquad\qquad v_2^z \sim \frac{F^z}{\rho z^2 U} \sim v_1^z\cdot\frac{1}{R}$$

and indeed $\vec{v}_2$, the part of the velocity generated by the pressure, is negligible for large enough $z$.

The diffusion process gives a simple picture for the shape of the wake. The size of the wake in $x, y$ is

$$a = \sqrt{\frac{4\nu z}{U}}$$

This is consistent with our earlier expression

$$\Delta y = \sqrt{z/R}$$

for the thickness of a laminar boundary layer.

Now I would like to continue to build the theory we will need to compute the lift. The surface integral over the surface $B$ in the lift formula is

$$\int_B dx\,dy\,v^y = \int dx \left\{ \int_{y_1}^{\infty} dy\,\frac{\partial\phi}{\partial y} + \int_{y_2}^{y_1} dy\,v^y + \int_{-\infty}^{y_2} dy\,\frac{\partial\phi}{\partial y} \right\}$$
$$= \int dx \left\{ -\phi(y_1) + \phi(y_2) + \int_{\text{wake}} dy\,v^y \right\}$$

where $y_1$ and $y_2$ are points just outside the wake region. We can represent the velocity potential difference $\phi(y_2) - \phi(y_1)$ equally well using any contour $C$ that runs from $y_1$ to $y_2$ through the region of potential flow.

$$\phi(y_2) - \phi(y_1) = \int_{y_1}^{y_2} d\vec{l}\cdot\vec{v}$$

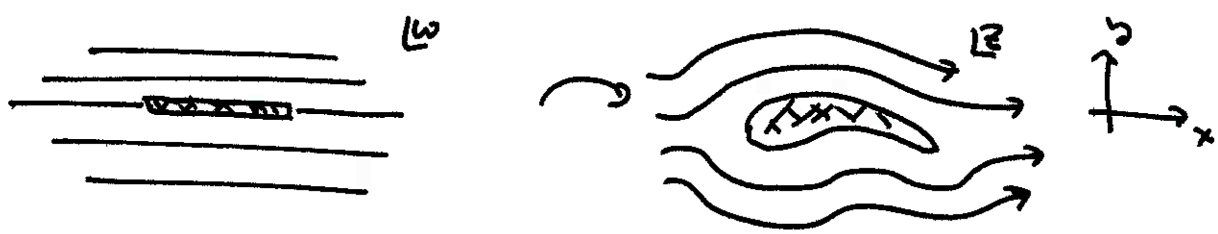

Adding back the short piece of contour that runs across the wake, we have

$$\phi(y_2) - \phi(y_1) = \oint_C d\vec{l}\cdot\vec{v} = \Gamma$$

where now $C_x$ is a closed contour that encircles the body at a fixed value of $x$. This is *Zhukovskii's theorem.*

This is a very weird formula. On most of the contour, $\vec{v} = \vec{\nabla}\phi$. However, if $\vec{v} = \vec{\nabla}\phi$ on the entire contour, and if $\phi$ were single-valued in the region away from the body, the lift would be zero. For a narrow wake, the short integral across the wake can be neglected. Then we can obtain lift only if the $\phi(\vec{x})$ that describes the potential flow outside the body is *not single-valued.*

Earlier in the course, we discussed a method based on complex analysis for finding 2-dimensional incompressible irrotational flows satisfying nontrivial boundary conditions. The method was to conformally map the complex plane with a slit along the real axis into an exterior region in the complex plane whose boundary had the shape of the object we were considering.




The corresponding equations are

$$\frac{dw}{dz} = v_x - i v_y$$

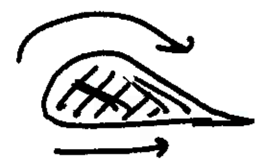
$$w = \phi + i\psi$$

$\phi$ = velocity potential
$\psi$ = stream function

where now I am writing $\phi$ for the velocity potential and $\psi$ for the stream function. This construction gives a $\phi$ that is single-valued as we go around the body. Thus, this method gives *zero* lift.

In elementary discussions of fluid mechanics, it is sometimes said that lift can be obtained from Bernoulli's theorem,

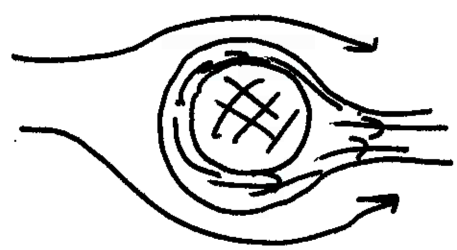
$$\frac{1}{2}v^2 + \frac{p}{\rho} = \text{const.}$$



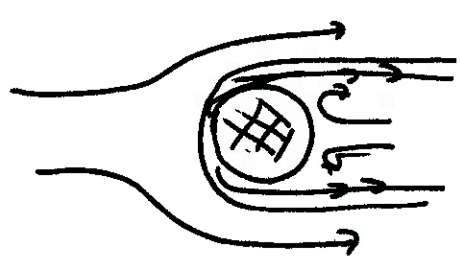
If we design an airplane wing, for example, to have larger $v$ above the wing than below, the pressure must be lower above the wing, and so we would expect to have lift. However, if we have potential flow everywhere, the flows above and below the wing must match together smoothly behind the wing. So if the flow is faster above than below on one part of the wing, it must be slower in another part, and the net effect of the pressures must cancel out.

Thus, *lift requires viscosity.* A region of nonzero vorticity behind the wing is essential. In a moment, I will perform an illustrative calculation of a the lift on a thin wing. First, though, I will make some general remarks on the shape of the wake and boundary layer regions as a function of Reynolds number.

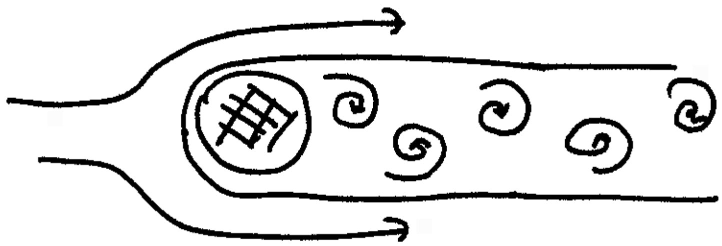
Consider the high Reynolds number flow around a thick body, for example, a cylinder. Please look again at the beautiful figures in Batchelor's book, Figs. 5.11.3 and 5. 11.4. The boundary layer begins tightly attached to the cylinder.



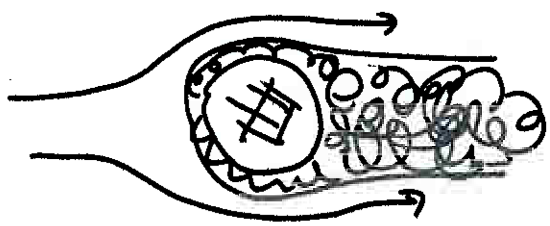
However, in the region of potential flow, the fluid velocity must decrease behind the cylinder. Then, by Bernoulli's theorem, the pressure must increase. For large enough $U$, this becomes a large pressure, and so the flow becomes unable to penetrate this region. At this point, the potential flow will no longer go inward to fill in the region behind the cylinder, and the boundary layer also changes its shape. It *separates* from the surface of the cylinder to fill the region that the potential flow has evacuated.



At still higher $U$, the flow in the boundary layer can drive a chain of vortices behind the cylinder



This is called the *von Karman vortex street.* At a still later point, the flow in the boundary layer develops instabilities and becomes turbulent



Measurements of the drag on a cylinder showw an interesting dependence on Reynolds number. We defined the drag coefficient $C_D$ by

$$C_D = \frac{\text{drag force}/\text{cm}}{\rho U^2 a}$$

where $a$ is the radius of the cylinder. This should only be a function of the Reynolds number

$$R = \frac{Ua}{\nu}$$

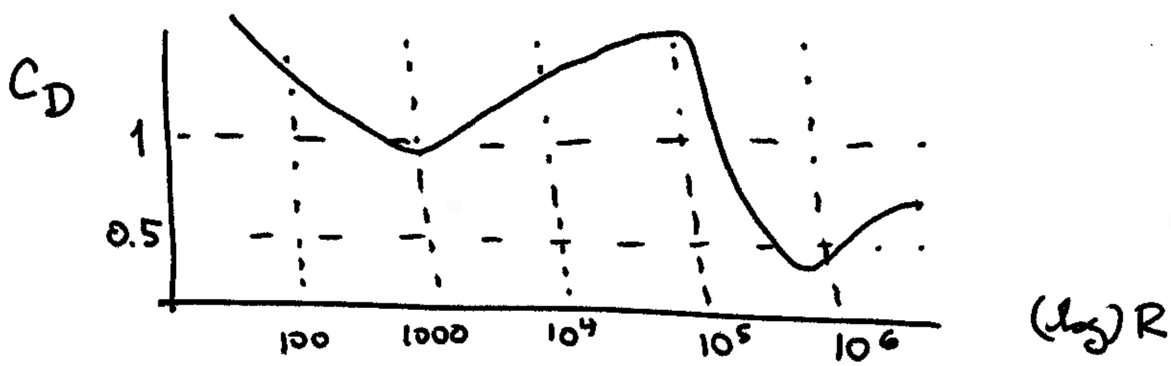
As I have discussed earlier in the course, for small Reynolds number, the drag force is estimated by dimensional analysis as

$$F = \rho \nu U \qquad \frac{g}{cm^3} \cdot cm^2/sec \cdot cm/sec = \frac{g\,cm/sec^2}{cm}$$

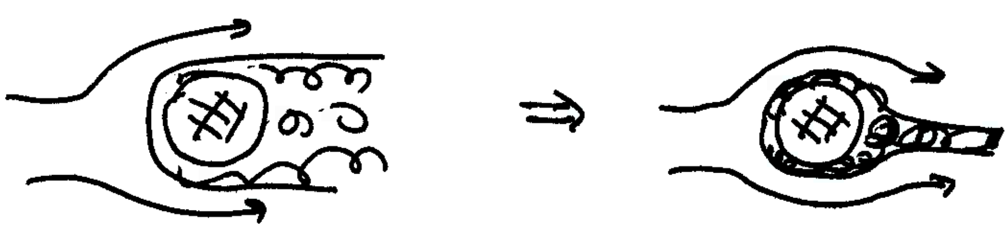
This is independent of $a$, which does not make sense. A detailed calculation shows that there is a logarithmic dependence on $a$. This gives

$$C_D = \frac{4\pi}{R} \frac{1}{(\log 4/R + \cdots)}$$

For larger values of $R$, the measured values of $C_D$ behave as follows:

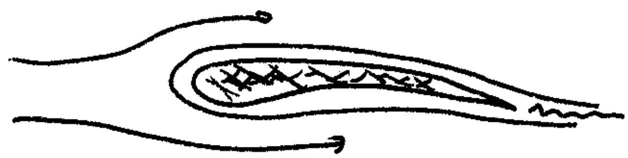



The increase in $C_D$ at $R \sim 10^3$ is the result of the transition explained above. The other feature of this curve is the sharp drop at $R \sim 10^5$. This is called the *drag crisis.* Prandtl gave the following explation for it. When the flow in the boundary layer switches from laminar to turbulent flow, the boundary layer becomes thicker and also more effective at transfering momentum. This makes it more resistance to separation from the cylinder. Thus, when the flow in the boundary layer becomes turbulent, the boundary layer snaps back to a position closer to the cylinder, and the wake region becomes smaller.

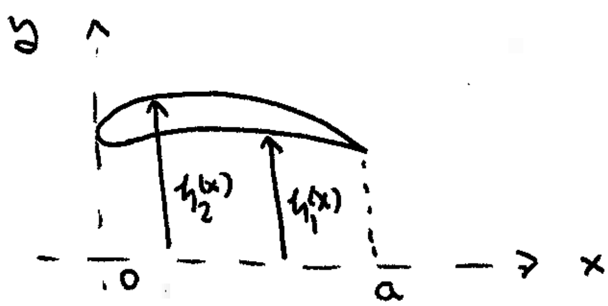


We have already seen that a thinner wake implies less drag.

To minimize the drag on a wing, we need to streamline the wing shape to avoid boundary layer separation as much as possible.



I will now present an illustrative calculation of the *lift* of a thin wing. I will assume that the wing is long in the direction into the paper, so that the flow can be treated as 2-dimensional. The method I will use is due to Keldysh and Sedov. This method uses tools from complex analysis, so I will relabel the coordinates as




As indicated, the position of the upper surface of the wing is $\zeta_2(x)$ and the position of the lower surface of the wing is $\zeta_1(x)$.

We need to find a flow around the wing of the form

$$\vec{V} = U\hat{x} + \vec{v}(x,y)$$

in which $\vec{v}$ is a potential flow except the wake region just behind the wing. For a very thin wing, we can represent $\vec{v}$ as a potential flow for which the potential has a *discontinuity* along a line just behind the wing.

The method of conformal mapping gives a potential flow for which the velocity is parallel to the wing at the surface and the velocities above and below the wing join together smoothly behind the wing. Keldysh and Sedov suggest that the correct situation for a high Reynolds number flow corresponds to a somewhat different boundary condition. They imagine that the wing is surrounded by a thin boundary layer. Outside this boundary layer, the $\hat{x}$ component of $\vec{V}$ is very close to $U$. Still, the flow must be parallel to the boundary, and so the $\hat{y}$ component of $\vec{V}$ must be adjusted to make the direction of the flow correct. This implies the boundary conditions

$$\text{above :} \quad v_y = U \frac{d\eta_2}{dx} \qquad 0 < x < a$$

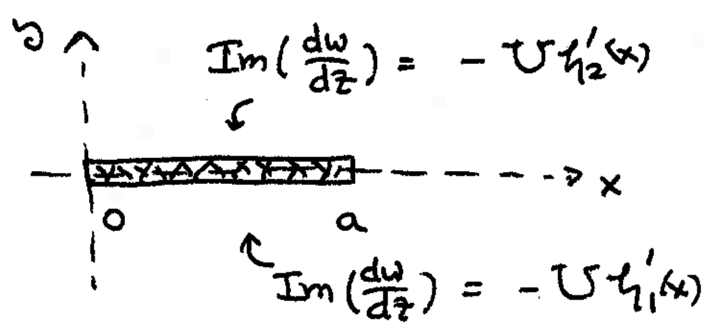
$$\text{below :} \quad v_y = U \frac{d\eta_1}{dx} \qquad 0 < x < a$$

with $v_y$ being continuous for $x < 0$ and for $x > a$. If the wing is thin, we can make the approximation that these conditions hold at $y = 0$. Then $v_y$ will have a discontinuity across a slit in the $(x, y)$ plane from $x = 0$ to $x = a$.

Now we can call on methods from the theory of analytic functions. Let $z = x + iy$. We would like to construct an function $\omega(z)$

$$\omega = \phi + i\psi \qquad \frac{dw}{dz} = v_x - iv_y$$

that is analytic in the complex $z$ plane away from a slit between $z = 0$ and $z = a$. The fuction $dw/dz$ should have the boundary values




and should go to zero as $z \to \infty$.

We would like to solve this problem for arbitrary functions $\zeta_1(x)$ and $\zeta_2(x)$. However, it is easiest to first consider separately the cases $\zeta_1(x) = \zeta_2(x)$ and $\zeta_1(x) = -\zeta_2(x)$. The general solution can be built from the solutions for these two cases.

Consider first the case $\zeta_-(x) = \zeta_2(x) = -\zeta_1(x)$. The solution in this case is

$$\frac{dw}{dz} = -\frac{U}{\pi}\int_0^a d\hat{x}\;\frac{\zeta'_-(\hat{x})}{\hat{x}-z}$$

To check this, note that, for $z = x + i\epsilon$,

$$\frac{1}{\hat{x}-z} = \mathcal{P}\,\frac{1}{\hat{x}-x} + i\pi\,\delta(\hat{x}-x)$$

so that

$$\operatorname{Im}\frac{dw}{dz} = U\,\zeta'_-(x)$$

just above the slit. Similarly, for $z = x - i\epsilon$,

$$\frac{1}{\hat{x}-z} = \mathcal{P}\,\frac{1}{\hat{x}-x} - i\pi\,\delta(\hat{x}-x)$$

so that

$$\operatorname{Im}\frac{dw}{dz} = -U\,\zeta'_-(x)$$

just below the slit. Finally, as $z \to \infty$,

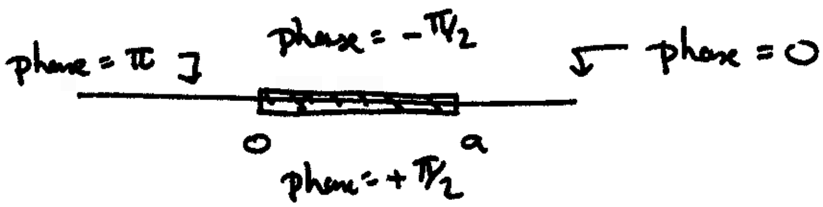
$$\frac{dw}{dz} \sim \frac{U}{\pi z} \int_0^a d\hat{x} \; \zeta_-'(\hat{x})$$

so all of the requirements are satisfied.

Now turn to the opposite case, $\zeta_+(x) = \zeta_2(x) = +\zeta_1(x)$. Here it is a little trickier to find the solution. We need to use the fact that the function

$$\sqrt{\frac{z}{z-a}}$$

changes sign across the cut from $z = 0$ to $z = a$. I define this function so that it has the phase




Notice that, with this choice,

$$\frac{dw}{dz}\sqrt{\frac{z}{z-a}}$$

has the opposite sign on two sides of the real axis at the cut, and so the previous solution applies to this function. Then

$$\frac{dw}{dz} \cdot \sqrt{\frac{z}{z-a}} = i\,\frac{U}{\pi} \int_0^a d\hat{x} \; \frac{\zeta_+'(\hat{x})}{\hat{x} - z} \sqrt{\frac{\hat{x}}{a-\hat{x}}}$$

For the general case of independent $\zeta_1(x)$ and $\zeta_2(x)$, the solution is a linear combination of these two forms. So we find

$$\frac{dw}{dz} = -\frac{U}{2\pi}\int_0^a d\hat{x}\,\frac{\zeta_2'(\hat{x})-\zeta_1'(\hat{x})}{\hat{x}-z}$$

$$+\,i\,\frac{U}{2\pi}\sqrt{\frac{z-a}{z}}\int_0^a d\hat{x}\,\frac{\zeta_2'(\hat{x})+\zeta_1'(\hat{x})}{\hat{x}-z}\sqrt{\frac{\hat{x}}{a-\hat{x}}}$$

We can compute $\omega(z)$ by integrating the expression on the right-hand size. However, this integral contains a logarithm and therefore a logarithmic branch cut extending to infinity. Thus, $\omega(z)$ and $\phi(x,y)$ are not single-valued in the region outside the slit.

The circulation on a contour $C$ around the slit is

$$I = \oint_C dx\,v_x + dy\,v_y = \mathrm{Re}\oint_C dz\,\frac{dw}{dz}$$


The function $d\omega/dz$ that we have constructed is analytic away from the slit, so we can push the contour outward and evaluate it using a contour at very large $|z|$. If $d\omega/dz$ has the asymptotic behavior

$$\frac{dw}{dz} \sim \frac{\gamma}{2\pi i z}$$

the it follows that

$$I = \mathrm{Re}\,\gamma$$

To evalute $\Gamma$ we only need to work out the asymptotic limit and identify $\gamma$ Taking the $z \to \infty$ limit of the expression on the previous page

$$\frac{dw}{dz} \sim \frac{U}{2\pi i z} \left\{ i \int_0^a d\hat{x} \, \left(\zeta_2'(\hat{x}) - \zeta_1'(\hat{x})\right) + \int_0^a d\hat{x} \, \left(\zeta_2'(\hat{x}) + \zeta_1'(\hat{x})\right) \sqrt{\frac{\hat{x}}{a-\hat{x}}} \right\}$$

we find

$$\gamma = U \cdot \left\{ i \int d\hat{x} \left(\zeta_2'(\hat{x}) - \zeta_1'(\hat{x})\right) + \int d\hat{x} \left(\zeta_2'(\hat{x}) + \zeta_1'(\hat{x})\right) \sqrt{\frac{\hat{x}}{a-\hat{x}}} \right\}$$

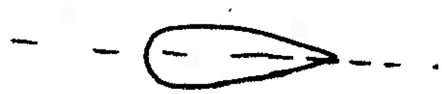
so that

$$\Gamma = U \cdot \int_0^a d\hat{x} \, \left(\zeta_2'(\hat{x}) + \zeta_1'(\hat{x})\right) \sqrt{\frac{\hat{x}}{a-\hat{x}}}$$

and, finally, the lift is

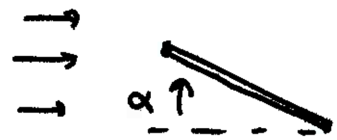
$$F^y = -\rho U^2 \int_0^a d\hat{x} \, \left(\zeta_2'(\hat{x}) + \zeta_1'(\hat{x})\right) \sqrt{\frac{\hat{x}}{a-\hat{x}}}$$

The pieces from the case $\zeta_2(x) = -\zeta_1(x)$ do not contribute to the lift. This makes sense, since the condition $\zeta_2 = -\zeta_1$ gives a symmetrical body



I can illustrate this formula by working out the example of a thin plate at a small angle of attack $\alpha$

$$\eta_1 = \eta_2 = \alpha\,(a - x)$$




$$\eta_1' = \eta_2' = -\alpha$$

We need the integral

$$\int_0^a dx \sqrt{\frac{x}{a-x}} = \frac{\pi}{2}\,a$$

Then

$$\Gamma = -\alpha \pi a U$$

and we obtain positive lift

$$F^y = +\pi\alpha \cdot \rho U^2 a$$

As a final topic in this lecture, I will discuss the shapes of turbulent wakes and boundary layers.

Consider first turbulent wakes. The shape of the wake is determined by the largest-scale motions in the turbulent flow.


To analyze this, I will follow the route that we used in discussing the Kolmogorov theory. I will make estimates using dimensional analysis, but without using the kinematic viscosity $\nu$ as a dimensionful parameter. The dimensionful quantities available are then the incoming fluid velocity $U$, and the distance $z$ along the flow. Let $v$ be the characteristic value of the velocity in the largest-scale turbulent eddies. This should depend on $U$ and $z$ but should also be a measure of the amount of turbulence in the flow.

Let $a$ be the width of the wake in the $(x, y)$ directions. The value of $a$ can change with $z$, increasing at an angle whose size is of order $v/U$

$$\frac{da}{dz} \sim \frac{v}{U}$$

Outside the wake, we have a region of potential flow in which the analysis at beginning of this lecture still applies. Thus, the force on the fixed body is still given by

$$F^i = -\rho U \int_B d^2x \; v^i$$

We can estimate the integral as $v$ times the cross sectional area of the wake, Then

$$F/\rho \sim U v a^2$$

The value of $F/\rho$ is fixed and independent of $z$, so this is an equation relating $v(z)$ to $a(z)$. It implies

$$\frac{da}{dz} \approx \frac{F}{\rho U^2 a^2}$$

This integrates to

$$a \approx \left(\frac{Fz}{\rho U^2}\right)^{1/3}$$

with

$$v \approx \left(\frac{F}{\rho}\frac{U}{z^2}\right)^{1/3}$$

It is interesting to compare the parametric shapes of a turbulent wake and a laminar wake

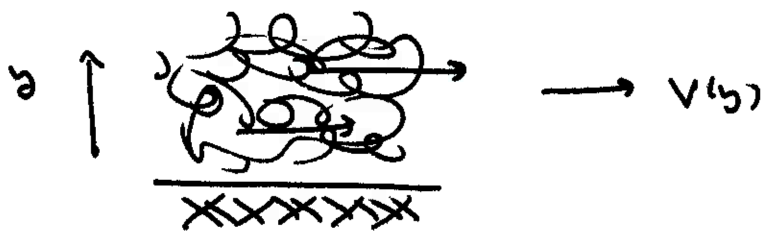
| turbulent wake | laminar wake |
| --- | --- |
| $a \sim z^{1/3}$ | $a \sim z^{1/2}$ |
| $v \sim \frac{1}{z^{2/3}}$ | $v \sim \frac{1}{z}$ |

The effective Reynolds number in the turbulent wake at a distance $z$ from the body is

$$R_{eff} = \frac{av}{\nu} \sim \frac{(F/\rho)^{2/3}}{\nu U^{1/3}} \frac{1}{z^{1/3}}$$

so the turbulence becomes less intense as $z$ increases. When $R_{eff}$ becomes small enough, the flow becomes smooth and the flow relaxes to a laminar flow. Beyond this point, the shape of the wake is determined by diffusion and $a \sim z^{1/2}$.

Next, we will consider the turbulent boundary layer. We can build up our understanding of this system by starting with turbulent flow over a flat plate.



The fluid acts on the plate with a drag force $f$ per unit area. Then the mean velocity of the fluid must increase as a function of the distance $y$ from the plate. We can work out this dependence using dimensional analysis

$$\frac{f}{\rho} \sim \frac{g\,cm/sec^2 / cm^2}{g/cm^3} \sim \frac{cm^2}{sec^2}$$

It is useful to write the characteristic quantity with units cm/sec

$$V_* = \sqrt{\frac{f}{\rho}}$$

Then, if we have fully developed turbulence that is indepedent of $\nu$ at large scales

$$\frac{dV}{dy} \sim \frac{V_*}{y}$$

This equation integrates to

$$V = \frac{V_*}{b} \log \frac{y}{c}$$

for some constants $b$, $c$.

The logarithm is cut off at a small value of $y$ where the distance to the plate is sufficiently small that the viscosity becomes relevant. This distance is estimated by

$$\frac{V_* y}{\nu} \sim 1$$

Then

$$V = \frac{V_*}{b} \left( \log \frac{y V_*}{\nu} + \mathrm{const} \right)$$

Landau and Lifshitz claim that this formula gives a good representation of experimental data with the parameter values

$$V = V_* \cdot \left[ 2.4 \log \frac{y V_*}{\nu} + 5.84 \right]$$

As an application of this formula, we can compute the relation between the pressure gradient and the flow velocity for a pipe in which the flow has become turbulent. Let $Q$ be the mass flow through the pipe per second and define $U$ by

$$\frac{Q}{\rho} = \pi a^2 U$$

where $a$ is the radius of the pipe. For Poiseuille flow

$$V(r) = V_0 \left(1 - \frac{r^2}{a^2}\right) \qquad \frac{Q}{\rho} = \frac{1}{2}\pi a^2 V_0$$

so

$$U = \frac{V_0}{2}$$

In our discussion of laminar flow in a pipe, we found that $V_0$ is related to the pressure gradient by

$$V_0 = \frac{a^2}{4\eta}\,\frac{\Delta p}{\ell}$$

The *resistance coefficient* of the pipe is defined by

$$\lambda = \frac{2a}{\frac{1}{2}\bar{\rho} U^2}\,\frac{\Delta p}{\ell}$$

For Poiseuille flow, $\lambda$ is given by

$$\lambda = \frac{32\nu}{aU} = \frac{64}{R}$$

where the Reynolds number is defined by

$$R = \frac{2aU}{\nu}$$

For turbulent flow, the largest-scale fluid velocity is

$$U \approx \frac{V_*}{b} \log\left(\frac{V_* a}{\nu}\right)$$

with $f$ given by

$$2\pi a f = \pi a^2 \frac{\Delta p}{\ell}$$

or

$$\frac{\Delta p}{\ell} = \frac{2}{a} f = \frac{2\rho}{a} V_*^2$$

Then

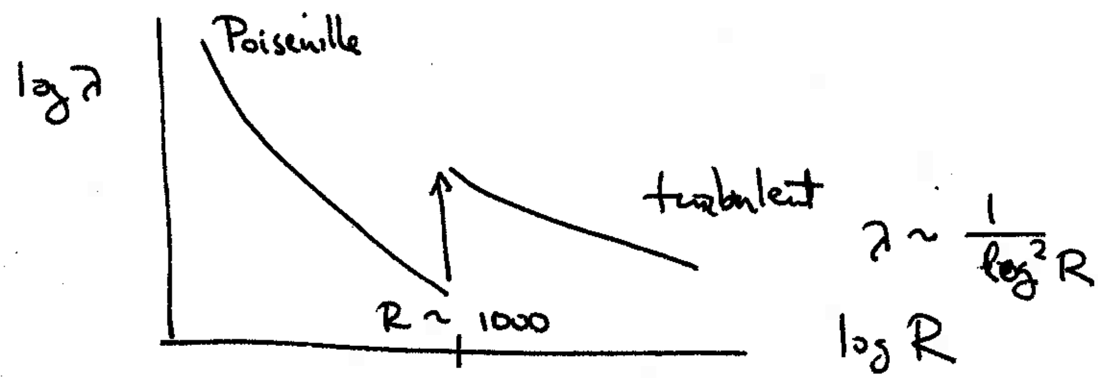
$$U \sim \frac{1}{b}\left(\frac{a}{2\rho}\frac{\Delta p}{\ell}\right)^{1/2} \log\left(\frac{a}{\nu}\left(\frac{a}{2\rho}\frac{\Delta p}{\ell}\right)^{1/2}\right)$$

$$\sim \frac{1}{2\sqrt{2}\,b}\left(\frac{4a}{\rho}\frac{\Delta p}{\ell}\right)^{1/2} \log\left[\frac{aU}{\nu}\left(\frac{4a}{\rho U^2}\frac{\Delta p}{\ell}\right)^{1/2}\right]$$

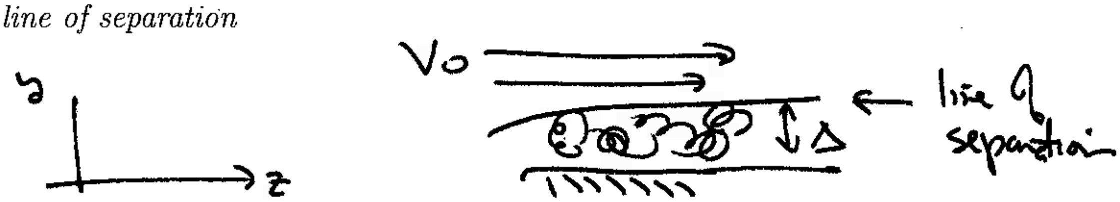
From this formula, we can compute

$$\lambda \approx \frac{8b^2}{\log^2(R\sqrt{\lambda})} = \frac{1.4}{\log^2(R\sqrt{\lambda})}$$

Thus, the behavior of $\lambda$ with Reynolds number has the form




We can apply the formulae for turbulent flow near a wall to compute the shape of a turbulent boundary layer. Consider a problem in which a very high Reynolds number flow with velocity $V_0$ streams across the surface of a body. As we discussed at the beginning of this lecture, the flow is sharply divided into two regions, such that, in the outer region, the flow is irrotational. The boundary between the two regions is called the *line of separation*



At this line of separation, the bulk velocity in the boundary layer should match the external veloocity $V_0$

$$V_0 = \frac{v_*}{b} \log\left( \frac{v_* \Delta}{\nu} \right)$$

The thickness of the boudary layer $\Delta$ should grow with $z$. This is compatible with the first requirement if the value of $v_*$ and the viscous drag generated by the boundary layer decrease with $z$.

As in the turbulent wake, the can use the eddy velocities in the turbulent layer to estimate the angle at which $\Delta$ grows with $z$

$$\frac{d\Delta}{dz} \sim \frac{v_*}{V_0}$$

The dependence of $v_*$ on $\Delta$ is only logarithmic, so to a first approximation we can consider $v_*$ as a constant and write

$$\Delta \sim \frac{v_*}{V_0} z$$

Then we can solve for $v_*$ in the equation

$$V_0 = \frac{v_*}{b} \log \frac{v_*^2 z}{\nu V_0}$$

to recover the logarithmic dependences as a correction.

$$v_* = \frac{bV_0}{\log \frac{v_*^2 z}{\nu V_0}} \cong \frac{bV_0}{\log \frac{b^2 V_0}{\nu}} + \cdots$$

This solution also gives us the value of $f = v_*^2 \rho$.

Define a local drag coefficient by

$$c = \frac{2f}{\rho V_0^2}$$

for a turbulent boundary layer, $c$ is given by

$$c = \frac{2v_*^2}{V_0^2}$$

Putting in the value of $v_*$ from above, we find

$$c \cong \frac{2b^2}{\log^2 (b^2 R_z)} \qquad R_z = \frac{V_0 z}{\nu}$$

More explicitly,

$$c \cong \frac{0.35}{\log^2 R_z}$$

The same analysis gives for the thickness of the boundary layer as a function of $z$

$$\Delta \sim \frac{z}{b \log R_z}$$

It is interesting to compare the formulae for turbulent and laminar boundary layers.

turbulent laminar

$$\Delta \sim 2.4 \frac{z}{\log\left(\frac{V_0 z}{\nu}\right)} \qquad \Delta \sim \sqrt{\frac{2\nu z}{V_0}}$$

$$C \sim \frac{0.35}{\log^2 \frac{V_0 z}{\nu}} \qquad C \sim 0.66 \sqrt{\frac{\nu}{V_0 z}}$$