Jan. 19

Viscous Flow

So far in this course, we have studied idealized fluids with no internal friction. I would now like to show how to include internal friction in the equations of fluid flow.

In the first lecture, we wrote the Euler equation as

$$\rho\left[\frac{\partial v^i}{\partial t} + (\vec{v}\cdot\vec{\nabla})\, v^i\right] = -\nabla^i p - \rho\nabla^i\Phi - \nabla_k T^{ik}_{\text{friction}}$$

Internal friction in the fluid should be described by an appropriate term in T^{ik}. To first this term, we need to write a constituitive equation relating T^{ik} to $\vec{v}$ and its derivatives. If $\vec{v}$ is slowly varying with respect to the atomic length scale, it will suffice to keep the terms with as few derivatives as possible that contain the effects we are looking for.

If the fluid moves as whole, $\vec{v}$ independent of $\vec{x}$, there should be no internal friction. Thus

$$T^{ik}_{\text{fr.}} \quad \text{should depend on} \quad \frac{\partial v^i}{\partial x^j}$$

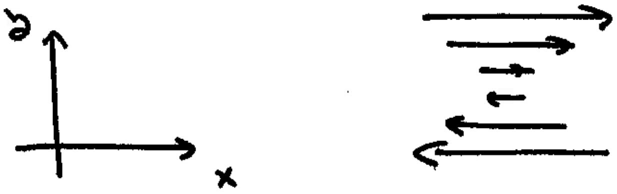
T^{ik} is a symmetric tensor so we need to build symmetric tensors from the components of $\partial v^i/\partial x^k$. There are two possible structures,

$$\Theta = \frac{\partial v^i}{\partial x^i} = \vec{\nabla}\cdot\vec{v}$$

the rate of expansion, and

$$\sigma^{ik} = \frac{1}{2}\left(\frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i}\right) - \frac{1}{3}\,\delta^{ik}\,\frac{\partial v^j}{\partial x^j}$$

the *rate of shear.* By construction, σ^{ik} is a *traceless* tensor. The term θ gives the trace part of $\partial v^i/\partial x^k$. A flow of the form



$$\sigma^{xy} = \sigma^{yx} > 0$$

has nonzero shear. In such a flow, the top layer of fluid exerts a friction force on the bottom layer, and vice versa.

The general form of the friction part of T^{ik}, linear in the velocity and containing the minimal number of derivatives, is then

$$T^{ik}_{fr} = -2\eta\,\sigma^{ik} - \zeta\,\theta\,\delta^{ik}$$

The internal friction in a fluid is called *viscosity.* Then η and ζ are the *coefficients of viscosity.* For an incompressible fluid, ζ is irrelevant, and so we usually call η the viscosity of the fluid. (Please notes that, in Batchelor's book, η is called μ.) It is useful to define the *kinematic viscosity*

$$\nu = \frac{\eta}{\rho}$$

For water, $v \approx 0.01\ \mathrm{cm}^2/\mathrm{sec}$ at room temperature.

The assumption that the internal friction part T^{ik} is linear in v defines a *Newtonian fluid.* This is a very typical behavoir, but there are exceptions. For example, cornstarch in water has a lower viscosity for sufficiently high shear (try it). Slurries, colloids, and other soft materials are often non-Newtonian, and the study of the material effect on the viscosity is a current research topic.

The contribution of viscosity to the Euler equation is

$$-\nabla_k T^{ik} = \nabla_k \left[2\eta\sigma^{ik} + \zeta\Theta\delta^{ik} \right]$$

$$= \eta\left(\nabla_k\nabla_i v^k + \nabla_k\nabla_k v^i - \tfrac{2}{3}\nabla_k\delta^{ik}\vec{\nabla}\cdot\vec{v}\right) + \zeta\nabla^i\vec{\nabla}\cdot\vec{v}$$

$$= \eta\nabla^2 v^i + \left(\tfrac{1}{3}\eta + \zeta\right)\nabla^i\vec{\nabla}\cdot\vec{v}$$

For an incompressible fluid, the second term is zero, and we find for the Euler equation

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi + \nu\nabla^2\vec{v}$$

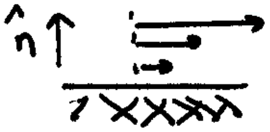
This is the *Navier-Stokes equation.* Most of this course will be devoted to solutions to this equation.

The Navier-Stokes equation is nonlinear and dissipative. Thus, it has all of the difficulties that appear in partial differential equations. There is no proof of existence and uniqueness theorems for the Navier-Stokes equation. In fact, this is one of the Millennium Problems. We are in a different world here from that of Maxwell's equations.

Because the Navier-Stokes equation is second-order in derivatives, we need to introduce an additional boundary condition with respect to the equation for ideal fluid flow. There, we imposed $\hat{n}\cdot\vec{v} = 0$ on the walls. For the Navier-Stokes equation, we need $\vec{v} = 0$ on the walls bounding the flow. That is, if there is friction, the first thin layer of fluid should stick to the wall. For a small element of fluid of size L, the friction stress is a force per unit area, thus giving a force of size L^2 while the mass of the fluid element is size L^3. As $L \to 0$, any small amount of friction can bring the fluid to rest.

If $\vec{v} = 0$ at the boundary but

$$\hat{n}\cdot\frac{\partial\vec{v}}{\partial\vec{x}} \neq 0$$



the fluid exerts a force on the wall and vice versa. The slice of fluid just next to the wall experiences a force from above of

$$+\ 2\eta\sigma^{ij}\hat{n}^j\cdot\text{Area} = \eta\frac{\partial v^i}{\partial x^j}\hat{n}^j\cdot\text{Area}$$

For a very thin slice, this must be balanced by a force from below. Thus, we have

$$\text{force/area of wall on fluid} = -\left[2\eta\sigma^{ij}\hat{n}^j + \zeta\Theta\hat{n}^i\right]$$

$$\text{force/area of fluid on wall} = +\left[2\eta\sigma^{ij}\hat{n}^j + \zeta\Theta\hat{n}^i\right]$$

In an incompressible fluid

$$\text{force/area of fluid on wall} = \eta\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right)\hat{n}^j$$

We must add this to the force due to pressure

$$-p\hat{n}^i$$

It is instructive to work out the effect of viscosity on the equation of energy conservation. For simplicity, I will consider an incompressible fluid and ignore gravity. (A complete analysis can be found in Landau and Lifshitz.)

We seek an equation of the form

$$\frac{\partial \rho_\epsilon}{\partial t} + \vec{\nabla}\cdot\vec{j}_\epsilon = (\text{dissipation})$$

Using the Navier-Stokes equation and $\rho =$ constant,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = \rho v^i\left[-(\vec{v}\cdot\vec{\nabla})\, v^i - \frac{1}{\rho}\nabla^i p + 2\,\frac{\eta}{\rho}\,\nabla_k \sigma^{ik}\right]$$

$$v^i(\vec{v}\cdot\vec{\nabla})v^i = \tfrac{1}{2}(\vec{v}\cdot\vec{\nabla})\, v^2 = \tfrac{1}{2}\vec{\nabla}(\vec{v}\, v^2) \quad \text{since } \vec{\nabla}\cdot\vec{v} = 0$$

Then

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = -\nabla^i\left[\rho v^i\, \tfrac{1}{2}v^2 + v^i p - 2\eta\, v^j \sigma^{jk}\right]$$
$$- 2\eta\,(\nabla^i v^j)\,\sigma^{ij}$$

This last term is

$$-\eta\left(\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j}\right)\frac{\partial v^j}{\partial x^i} = -\tfrac{1}{2}\eta\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right)^2$$

Then this equation has the form

$$\frac{\partial}{\partial t}\rho_\epsilon + \vec{\nabla}\cdot\vec{j}_\epsilon = -\frac{1}{2}\eta\left(\frac{\partial v^i}{\partial x^j}+\frac{\partial v^j}{\partial x^i}\right)^2$$

with

$$\rho_\epsilon = \frac{1}{2}\rho v^2 \qquad j_\epsilon^i = \rho v^i\left(\frac{1}{2}v^2+\frac{P}{\rho}\right) - 2\eta\sigma^{ij}v^j$$

Finally, the nonconservation of energy is given by

$$\frac{d}{dt}\int d^3x\,\rho_\epsilon = -\frac{1}{2}\eta\int d^3x\left(\frac{\partial v^i}{\partial x^j}+\frac{\partial v^j}{\partial x^i}\right)^2$$

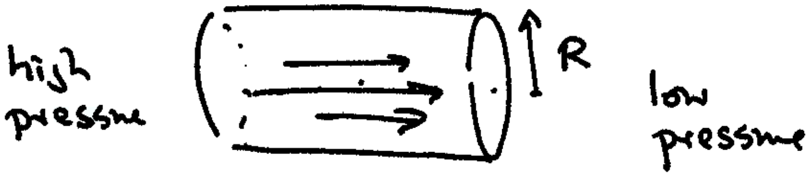
This is not actually a loss of energy but rather the conversion of mechanical energy in the fluid into heat. I will write a more explicit equation for this aspect of the energy flow later in the course. By the second law of thermodynamics, mechanical energy should decrease in a flowing fluid. Thus, we require

$$\eta > 0$$

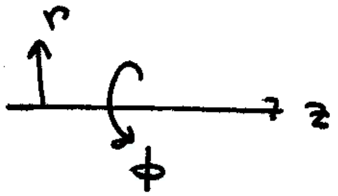
A similar analysis for a compressible fluid requires also

$$\zeta > 0$$

Let's now analyze some explicit examples of flows where viscosity is important. The simplest such flow is flow in a pipe, called *Poiseuille flow*,



Use the cylindrical coordinates



For steady flow, the Navier-Stokes equation becomes

$$(\vec{v}\cdot\vec{\nabla})\,\vec{v} = -\frac{\vec{\nabla}p}{\rho} + \nu\,\nabla^2\vec{v}$$

We want to find a solution with $\vec{v}$ parallel to $\hat{z}$ and depending only on x, y or on r, ϕ. In such a case, the situation is uniform along the pipe, so the pressure *gradient* should also be independent of z. Write, then,

$$\frac{dp}{dz} = -G = \text{const.} = \frac{\Delta p}{\ell}$$

The $\hat{x}$, $\hat{y}$ components of the Euler equation are

$$0 = -\frac{\vec{\nabla} p}{\rho} + 0$$

Then p is *constant* on a cross section of the pipe. Also,

$$(\vec{\nabla} \cdot \vec{\nabla}) \vec{v} = 0$$

since $\vec{v}$ is parallel to $\hat{z}$ but $\vec{v}$ does not depend on z. Then the $\hat{z}$ component of the Navier-Stokes equation becomes

$$0 = \frac{1}{\rho} G + \nu \nabla^2 v_z$$

Putting in the constant gradient of p, we find

$$\nabla^2 v_z = -\frac{G}{\rho \nu} = \text{const.}$$

To solve this equation, $v_z(x, y)$ must have a quadratic dependence on x, y. The solution, subject to the boundary condition

$$v_z(r = R) = 0$$

is

$$V_z = V_0\left(1 - \frac{x^2+y^2}{R^2}\right) = V_0\left(1 - \frac{r^2}{R^2}\right)$$

Then

$$\nabla^2 V_z = -\frac{4V_0}{R^2} = -\frac{G}{\eta}$$

so that

$$V_0 = \frac{R^2}{4\eta}\,\frac{\Delta p}{\ell}$$

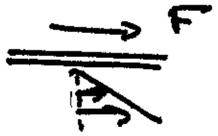
The mass flow through the pipe, in g/sec, is given by

$$\begin{aligned} Q &= 2\pi\rho \int_0^R dr\; r\; v_z(r) \\ &= 2\pi\rho\, V_0 \int_0^R dr\; r\,(1 - r^2/R^2) \\ &= 2\pi\rho\, V_0 \left(\tfrac{1}{2} - \tfrac{1}{4}\right) R^2 \end{aligned}$$

so that finally

$$Q = \frac{\pi}{2}\rho V_0 R^2 = \frac{\pi}{8}\,\frac{R^4}{\nu}\,\frac{\Delta p}{\ell}$$

It is interesting to compute the force that the water exerts on the pipe. The force per unit area on the interior walls of the pipe is



$$-\eta \frac{\partial v^2}{\partial r}\bigg|_{r=R} = +\eta V_0 \frac{2}{R}$$

The total force is

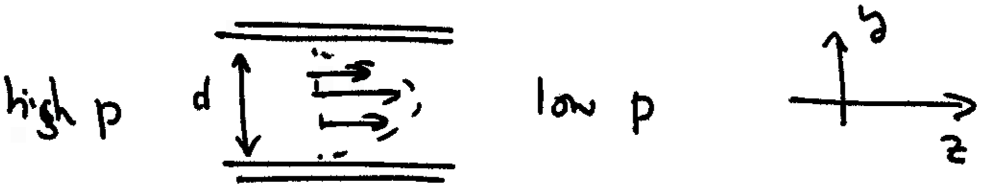
$$F = (2\pi \ell R)\cdot \eta V_0 \frac{2}{R} = 4\pi V_0\, \ell \cdot \eta$$

This is just

$$F = \pi R^2 \cdot \Delta p$$

that is, in a steady state, the water exactly transmits to the pipe the total force that is applied to it by the pressure gradient.

For later reference, I would like to write the similar formulae for 2-dimensional Poiseuille flow, that is, viscous flow between parallel plates.



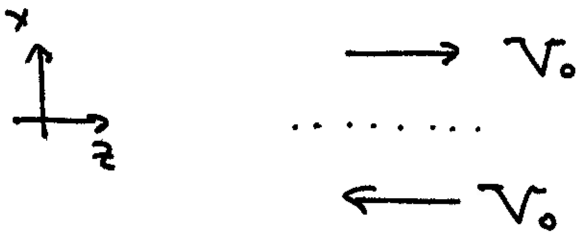
Again, the pressure gradient should be constant and the velocity should be in the $\hat{z}$ direction. Now $\vec{v}$ depends only on y. It obeys the equation

$$\nu \frac{d^2}{dy^2} v_z = - \frac{G}{\rho}$$

The solution is

$$v_z = 4V_0 \frac{y(d-y)}{d^2} \qquad V_0 = \frac{G d^2}{8\eta}$$

Now I would like to consider a simple time-dependent problem involving viscosity. Consider an initial state with a discontinuity in velocity between two layers of fluid. This is not a very realistic initial condition, for reasons we will discuss later in the course, but it is a place to start.



$$\vec{V}(x,0) = \begin{cases} V_0 \hat{z} & x > 0 \\ -V_0 \hat{z} & x < 0 \end{cases}$$

We can look for a solution in which $\vec{v}$ remains parallel to $\hat{z}$ and is only a function of x and t.

Applying the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}$$

we find again that, because gradients of $\vec{v}$ are orthogonal to $\hat{z}$, that the term with $(\vec{v}\cdot\vec{\nabla})\vec{v}$ is zero. The $\hat{x}$ component of the equation is

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial x} p + 0 \quad \text{or} \quad \frac{\partial p}{\partial x} = 0$$

so there is no effect from the pressure. The equation thus reduces to

$$\frac{\partial v_z}{\partial t} = \nu \nabla^2 v_z$$

This is just the diffusion equation, with ν in the place of the diffusion constant.

Here is another way to look at this result. Go back to the Navier-Stokes equation and assume only that the effects of the velocities is small, so that the nonlinear term with $(\vec{v}\cdot\vec{\nabla})\vec{v}$ is negligible compared to the viscous damping term. Then, taking the curl of the equation, we find

$$\frac{\partial}{\partial t}\vec{\omega} = \nu \nabla^2 \vec{\omega}$$

So, ν is more properly the *diffusion constant of vorticity.* Note that ν has the units of diffusion constant, cm^2/sec.

In any case, we can now apply the solution of the diffusion equation given in the first lecture. Here we have diffusion in one dimension. Then, the time-dependent solution is given in terms of the initial condition for v_z as

$$v_z(x,t) = \int dx' \frac{1}{[4\pi\nu t]^{1/2}} e^{-(x-x')^2/4\nu t} \quad v_z(x',0)$$

Now we can compute the velocity distribution for $x > 0$. The initial condition is

$$V_z(x',0) = V_0 - 2V_0\,\theta(-x')$$

Integrating the constant V_0 with the Gaussian gives back just a constant velocity V_0. Then

$$v_z(x,t) = V_0 - 2V_0 \int_{-\infty}^{0} dx' \frac{1}{[4\pi\nu t]^{1/2}}\, e^{-(x-x')^2/4\nu t}$$

Let

$$\omega = \frac{(x-x')}{\sqrt{4\nu t}}$$

Then

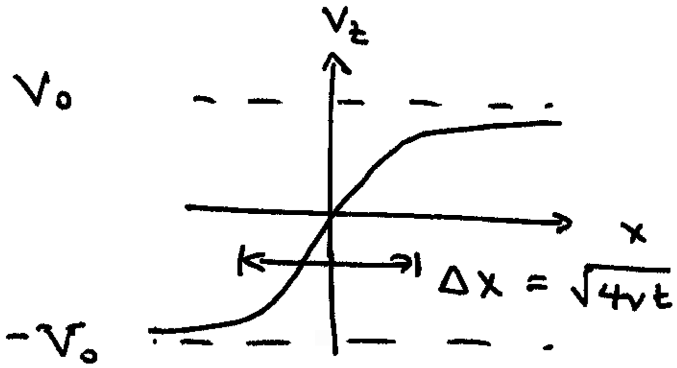
$$v_z(x,t) = V_0 - 2V_0 \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4\nu t}}^{\infty} d\omega\, e^{-\omega^2}$$

or, since

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} d\omega\, e^{-\omega^2} = 1$$

we find for the final result

$$V_z(x,t) = \frac{2V_0}{\sqrt{\pi}} \int_0^{x/\sqrt{4\nu t}} dw\, e^{-w^2} = V_0\, \mathrm{erf}\left(\frac{x}{\sqrt{4\nu t}}\right)$$



After a time t, the thickness of the transition region has increased to

$$\Delta x = \sqrt{4\nu t}$$

The general analysis given here applies to any initial condition with vv parallel to $\hat{z}$ and depending only on x. Thus, if the initial condition has a transition region of thickness ℓ, we find the same solution given above when

$$\sqrt{4\nu t} \gg \ell$$

A similar analysis can be used to describe the startup of flow in a pipe when pressure is applied to the fluid.

We are still discussing incompressible fluids and fluid motions much less than the speed of sound, so if a pressure gradient is suddenly applied to fluid in a pipe, the fluid will suddenly start to move, uniformly in the whole pipe. A flow parallel to $\hat{z}$ and uniform in z will again obey the equation

$$\frac{\partial v_z}{\partial t} = \frac{G}{\rho} + \nu \nabla^2 v_z$$

This equation has a stationary solution that is just Poiseuille flow

$$v_P = \frac{R^2}{4\eta} G \left(1 - r^2/R^2\right)$$

Then write

$$v_z(t) = v_P - W(t)$$

The time-dependent part of the solution obeys

$$\frac{\partial W}{\partial t} = \nu \nabla^2 W$$

with boundary conditions

$$w(t=0) = V_p \qquad w(r=R,t) = 0$$

This is another diffusion problem. If w is symmetric about the center of the pipe,

$$\frac{\partial w}{\partial t} = \nu \left(\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \right) w$$

From electrostatics or quantum mechanics, we know how to deal with this equation. The eigenvectors of the operator on the left are Bessel functions,

$$-\left(\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \right) J_0(kr) = k^2 \, J_0(kr)$$

The boundary condition in r is imposed by choosing k so that

$$k_n = \frac{z_n}{R} \qquad z_n = n^{\text{th}} \text{ zero of } J_0(z)$$

Then, we can expand

$$w(r,t) = \frac{R^2}{4\eta} G \sum_n A_n(t) \, J_0\!\left(z_n \frac{r}{R}\right)$$

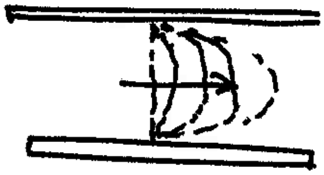
The coefficients $A_n(t)$ satisfy

$$\frac{\partial A_n(t)}{\partial t} = -\nu k_n^2 A_n(t)$$

Finally, we find the solution

$$v_z(r,t) = \frac{R^2}{4\eta}\frac{\Delta p}{\ell}\left[1 - \frac{r^2}{R} - \sum_n A_n J_0\left(z_n \frac{r}{R}\right) e^{-\nu k_n^2 t}\right]$$

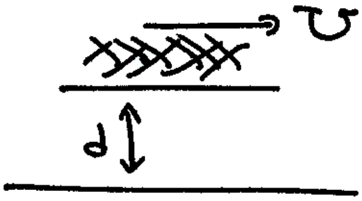
The initial values of the A_n are found by projecting the intial velocity distribution in the standard way. Once these values are established, the higher the value of n, the faster the relaxation. The functions with higher n are more important near the boundary at $r = R$, so the velocity diffuses in from the wall,

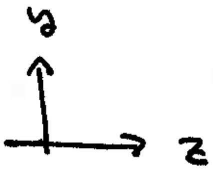


The next example that I will consider is the theory of *lubrication*, studied in the 19th century by Reynolds, whom we will meet later as the author of a more general theoretical idea in fluid flow. If we want to move a block across a surface, we put some oil under the block to replace solid-on-solid sliding friction by friction in the fluid,

If the block is at rest, it will squeeze the fluid out. However, if the block is in motion, it can set up a situation in which the fluid under the block is at high pressure, and this pressure can levitate the block. Here is the analysis:

First, consider the situation in which the block is exactly level and moving with constant velocity U.





The equations for this situation are the same as for 2-dimensional Poiseuille flow, but with a different boundary condition. If there is no pressure gradient, the fluid velocity will be

$$V_z = U \frac{y}{d}$$

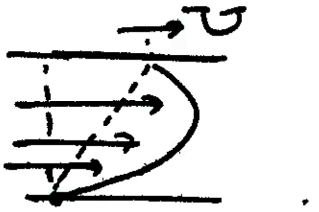
to give the correct fluid velocity on each surface. If there is a pressure gradient

$$\frac{dp}{dz} = -G$$

the solution is

$$v_z = \frac{G}{2\eta}\, y(d-y) + U\,\frac{y}{d}$$

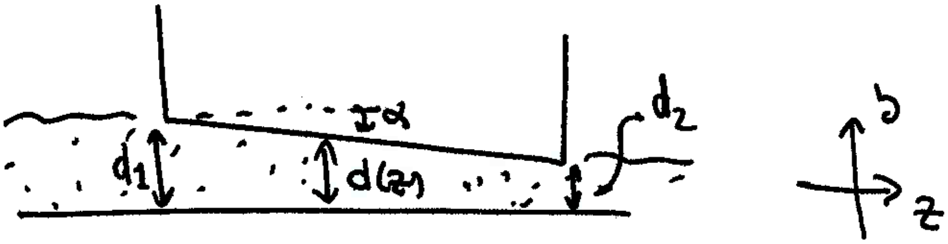
For $G > 0$, the form of the velocity field is



so as G increases, we also find a larger tangential force on the block.

There is still one ingredient missing. The fluid pressure must be equal to atmospheric pressure, p_0, on both the front and back sides of the block. This seems to give the situation that $p = p_0$ everywhere. There is no pressure gradient and no substantial levitating force, though there is a small tangential force.

Now consider the situation in which the block is tilted by a small angle α.



The thickness of the fluid layer now varies with x as

$$d(z) = d_1 - \alpha z$$

I claim that we can now have a solution with $p = p_0$ at both ends of the block but $p > p_0$ and even large in the center.

If α is small, $v_z(y)$ is given approximately by the expression above, with d replaced by $d(z)$, the local height of the channel. The mass flow Q past a line of fixed z is given by

$$\frac{Q}{\rho} = \int_0^d dy\, v_z = \frac{G\,d^3}{12\eta} + U\frac{d}{2}$$

The fluid is incompressible, so Q/ρ must be constant, that is, independent of z. So we can solve for G,

$$G = -\frac{dp}{dz} = -6\eta\left(\frac{U}{d^2} - \frac{2Q}{d^3}\right)$$

Put in $d = d_1 - \alpha z$; then

$$p - p_0 = \int_0^z dz\,(-G) = -\int_{d_1}^{d} \frac{d(d)}{\alpha}\, 6\eta\left(\frac{U}{d^2} - \frac{2Q}{d^3}\right)$$

$$= \frac{6\eta}{\alpha}\left[U\left(\frac{1}{d} - \frac{1}{d_1}\right) - Q\left(\frac{1}{d^2} - \frac{1}{d_1^2}\right)\right]$$

For a particular value of $d = d_2$, the pressure returns to $p = p_0$.

$$U\left(\frac{1}{d_2} - \frac{1}{d_1}\right) = Q\left(\frac{1}{d_2^2} - \frac{1}{d_1^2}\right)$$

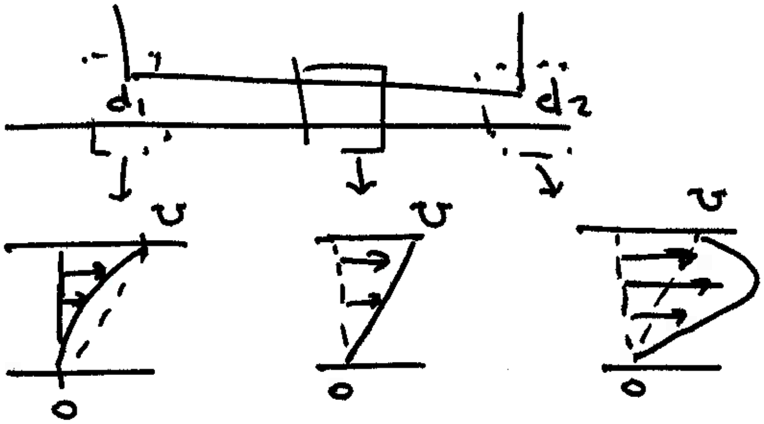
$$Q = U\left(\frac{d_1 d_2}{d_1 + d_2}\right)$$

Then

$$p - p_0 = 6\eta \frac{U}{\alpha} \frac{(d_1 - d)(d - d_2)}{d^2 (d_1 + d_2)}$$

There is a pressure maximum at an interior point. Naturally, to achieve smooth motion of the block at a fixed U, we adjust α so that the pressure on the bottom surface of the block is just enough to hold it up.

Here is a sketch of the fluid flow underneath the block at various positions:



It is interesting to compute the forces on the block. The normal force F_y is due to pressure. Then

$$F_y = \int dx\,(p - p_0) = \frac{6\eta U}{\alpha^2}\left[\log \frac{d_1}{d_2} - 2\left(\frac{d_1 - d_2}{d_1 + d_2}\right)\right]$$

The result is proportional to α^2. The second factor of α comes from converting from an integral over z to an integral over $d(z)$. The tangential force is due to the viscous stress

$$F_z = -\int dz\, \eta \frac{\partial v_z}{\partial y}\bigg|_{y=d}$$

$$F_z = -\int dz \; \eta \left(U \frac{1}{d} - \frac{G}{2\eta} d \right)$$

$$= -\frac{2\eta U}{\alpha} \left(3 \; \frac{d_1 - d_2}{d_1 + d_2} - \log \frac{d_1}{d_2} \right)$$

The *coefficient of friction* is

$$\left| \frac{F_z}{F_y} \right| = \alpha \cdot f(d_1/d_2)$$

Notice that U cancels out, and, more surprisingly, that η cancels out. The coefficient of friction in lubrication is independent of the viscosity of the fluid. But remember that oil is *more* viscous than water, with $\nu \sim 4$ cm^2/sec for motor oil. What is important about the lubricant is that it be viscous enough that we stay in the region where the approximation that vicosity dominates over nonlinear effects is a good one.