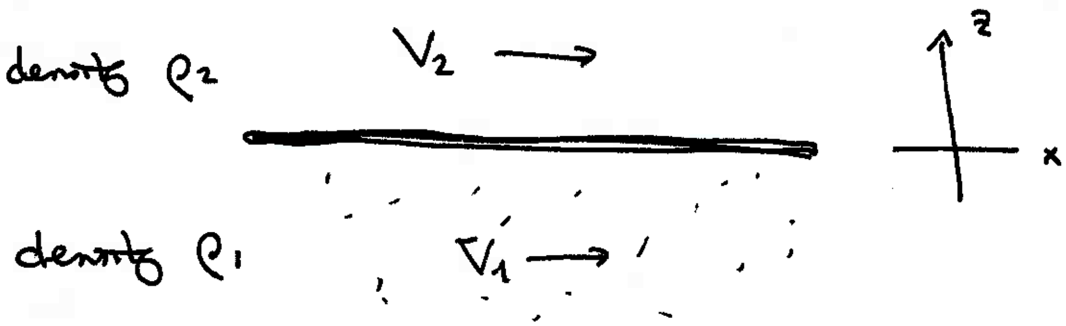
Jan. 28

# Fluid Instabilities

In the previous lecture, we saw that, as we go from low to high Reynolds number, a fluid flow can develop a *boundary layer* that concentrates shear in a thin layer than shrinks as the Reynolds number grows. This is only the simplest pathology of the Navier-Stokes equation that is encountered as the Reynolods number is increased. In this lecture, I will discuss another one, the propensity of stationary, laminar flows to become unstable as at high Reynolds number.

The simplest example of a fluid instability is the instability of an interface between two fluids with different densities or velocities. To understand this, we can analyze the following problem:



with the velocities $V_1$ and $V_2$ and the densities $\rho_1$ and $\rho_2$ arbitrary. Earlier in the course, we studied a problem like this with equal densities on the two sides, assuming that the flow was confined to the $\hat{x}$ direction and studying the evolution of the velocities due to diffusion of vorticity. Now, I would like to consider the opposite limit, setting $\nu = 0$ so all effects are due to convection. For $\nu = 0$, the situation with a sharp boundary shown in the diagram is a solution to the Euler equations. However, it might not be a *stable* solution.

There is another special case of this problem that we have studied in an earlier lecture, the problem of gravity waves on a surface. That is the limit $V_1 = V_2 = 0$ of this problem. Let's now analyze the general case.

The initial situation is

$$z > 0 \qquad \vec{v} = V_2 \hat{x} \qquad \rho = \rho_2 \qquad p = p_0 - g\rho_2 z$$

$$z < 0 \qquad \vec{v} = V_1 \hat{x} \qquad \rho = \rho_1 \qquad p = p_0 - g\rho_1 z$$

This initial condition has $\vec{\omega} = \vec{\nabla} \times \vec{v} = 0$ for $z > 0$ and for $z < 0$. However, there is a delta-function of vorticity at the interface.

In inviscid 2-dimensional flow, we saw in the previous lecture that

$$(\vec{v} \cdot \vec{\nabla})\, \vec{\omega} = 0$$

that is, vorticity is constant along streamlines. So, even if we make a localized perturbation, as long as $\omega = 0$ far to the left above and below the interface, we will have $\omega = 0$ in the whole region. Then we can use $\vec{\nabla} \times \vec{v} = 0$ and represent the flows with a velocity potential, which I will call $\phi$ in this problem. Write

$$z > 0 \qquad \vec{v} = \vec{\nabla} \phi_2$$

$$z < 0 \qquad \vec{v} = \vec{\nabla} \phi_1$$

with

$$\nabla^2 \phi_2 = \nabla^2 \phi_1 = 0$$

The zeroth-order solution is

$$\phi_2 = V_2 x + \delta \phi_2$$

$$\phi_1 = V_1 x + \delta \phi_1$$

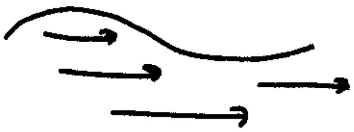
Call the position of the interface $z = \zeta(x,t)$, as in our earlier discussion. I will analyze the problem to linear order in $\delta\phi_1$, $\delta\phi_2$, $\zeta$.

The first thing we need to do is to find the boundary conditions at the interface that link $\phi_1$, $\phi_2$, and $\zeta$. In the frame in which the fluid is at rest, the vertical velocity of a point on the interface is $v_z = \partial\zeta/\partial t$. In a general frame, the vertical velocity is given by

$$\frac{D}{Dt}\zeta = \frac{\partial \zeta}{\partial t} + \vec{v}\cdot\vec{\nabla}\zeta$$

This allows that, if the interface is moving with the fluid

$$\zeta(x,t) = \zeta(x-vt)$$



the vertical component of velocity will be zero as required. On the other hand, the vertical velocity just away from the interface is $v_z = \partial\phi/\partial z$. Then, at the top of region 1,

$$\left.\frac{\partial \phi_1}{\partial z}\right|_{z=\zeta} = \frac{\partial \zeta}{\partial t} + \vec{v}_1\cdot\vec{\nabla}\zeta$$

and at the bottom of region 2,

$$\left.\frac{\partial \phi_2}{\partial z}\right|_{z=\zeta} = \frac{\partial \zeta}{\partial t} + \vec{v}_2\cdot\vec{\nabla}\zeta$$

We need a third equation to constrain the three unknowns at the interface; this is the continuity of the pressure,

$$p_2(x, z=\eta, t) = p_1(x, z=\eta, t)$$

In each of the two regions, we can use Bernoulli's theorem to constrain the pressure and the velocity potential,

$$\left[ \frac{\partial \phi_2}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi_2)^2 + \frac{p_2}{\rho_2} + gz \right] = C_2$$

$$\left[ \frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi_1)^2 + \frac{p_1}{\rho_1} + gz \right] = C_1$$

To evaluate the constants $C_1$, $C_2$, we can imagine that the perturbation is localized in $x$ and go to the unperturbed region far to the left or right. Here $p = -\rho g z$, so

$$C_1 = \frac{1}{2}V_1^2 \qquad C_2 = \frac{1}{2}V_2^2$$

Then the values of the pressure at the interface are

$$p_2 = \rho_2 \left[ \frac{1}{2}V_2^2 - \frac{1}{2}(\vec{\nabla}\phi_2)^2 - \frac{\partial \phi_2}{\partial t} - g\eta \right]$$

$$p_1 = \rho_1 \left[ \frac{1}{2}V_1^2 - \frac{1}{2}(\vec{\nabla}\phi_1)^2 - \frac{\partial \phi_1}{\partial t} - g\eta \right]$$

These two quantities must be equal. Expanding to first order in small quantities according to the formulae above the constraint is

$$\rho_1 [ V_1 \frac{\partial}{\partial x} \delta\phi_1 + \frac{\partial}{\partial t} \delta\phi_1 + g\zeta )$$
$$= \rho_2 [ V_2 \frac{\partial}{\partial x} \delta\phi_2 + \frac{\partial}{\partial t} \delta\phi_2 + g\zeta ]$$

while the constraints between $\zeta$ and the bulk velocities above and below give

$$\frac{\partial}{\partial z}\delta\phi_2 = \frac{\partial\zeta}{\partial t} + V_2 \frac{\partial}{\partial x}\zeta$$

$$\frac{\partial}{\partial z}\delta\phi_1 = \frac{\partial\zeta}{\partial t} + V_1 \frac{\partial}{\partial x}\zeta$$

Now I would like to introduce a notation for small deviations that I will follow throughout this lecture and in the rest of the course. The problem is translation invariant in $x$ and $t$, so it is convenient to look at the Fourier components in these variables. To linear order, we can study the dynamics of a single Fourier component. Write this as

$$\zeta(x,t) = \zeta\, e^{-i\omega t + ikx}$$
$$\delta\phi_2(x,z,t) = \phi_2(z)\, e^{-i\omega t + ikx}$$
$$\delta\phi_1(x,z,t) = \phi_1(z)\, e^{-i\omega t + ikx}$$

To repeat, I will use the notation for the bulk quantity, e.g. $\phi$ also to represent the amplitude of the Fourier component that I have picked out for study. In this context, $\phi$ is a complex-valued quantity. This allows a transparent and streamlined notation, as long as you keep the underlying concepts clearly in mind. I hope it does not cause too much confusion.

The velocity potentials $phi_1$ and $\phi_2$ are harmonic: $\nabla^2\phi_i = 0$. Using the notation just introduced, we can write the two pde's as

$$\nabla^2 \phi_2 = 0 \quad \Rightarrow \quad \left(-k^2 + \frac{d^2}{dz^2}\right) \phi_2(z) = 0$$

$$\nabla^2 \phi_1 = 0 \quad \Rightarrow \quad \left(-k^2 + \frac{d^2}{dz^2}\right) \phi_1(z) = 0$$

We can solve for the $z$-dependence using the boundary condition that the perturbation should go to zero for $z \to -\infty$ and for $z \to \infty$. This gives

$$\phi_1(z) = \phi_1 \, e^{kz} \qquad \qquad \phi_2(z) = \phi_2 \, e^{-kz}$$

Now we only need to solve the three matching conditions at the boundary. In the notation of the Fourier coefficients, these are

$$\rho_1 \left( ikV_1 \phi_1 - i\omega \phi_1 + g\zeta \right) = \rho_2 \left( ikV_2 \phi_2 - i\omega \phi_2 + g\zeta \right)$$

$$-k \, \phi_2 = (-i\omega + ikV_2) \, \zeta$$

$$+k \, \phi_1 = (-i\omega + ik \, V_1) \, \zeta$$

Then

$$\phi_2 = +i \, \frac{\omega - V_2 k}{k} \, \zeta$$

$$\phi_1 = -i \, \frac{\omega - V_1 k}{k} \, \zeta$$

Plugging these values into the third equation and eliminating $\phi_1$ and $\phi_2$ in favor of $\zeta$, we find

$$\rho_1(\omega - V_1 k)^2 + \rho_2(\omega - V_2 k)^2 = (\rho_1 - \rho_2) g k$$

This equation gives $\omega$ in terms of $k$, the dispersion relation of the surface waves.

We can now examine this result for several special cases. First, consider $V_1 = V_2 = 0$, $\rho_2 = 0$. Here we recover our earlier result for gravity waves on a surface

$$\omega^2 = gk$$

Next, consider $V_1 = V_2 \neq 0$. This gives

$$\omega = Vk \pm \sqrt{gk} \qquad V = V_1 = V_2$$

which is just the Galilean boost of the previous situation.

Next, consider $V_2 > V_1$, with $\rho_1 = \rho_2 = \rho$. Here we find

$$\omega^2 - (V_1 + V_2)\omega k + \tfrac{1}{2}(V_1^2 + V_2^2) k^2 = 0$$

$$\left[\omega - \frac{V_1 + V_2}{2} k\right]^2 + \left(\frac{V_1 - V_2}{2}\right)^2 k^2 = 0$$

so that finally

$$\omega = \frac{V_1+V_2}{2} \pm i \, \frac{V_1-V_2}{2} \, k$$

For real $k$, that is, fixed wavelength of the perturbation, the frequency is imaginary. The solution with a positive imaginary part, inserted into the time-dependent factor

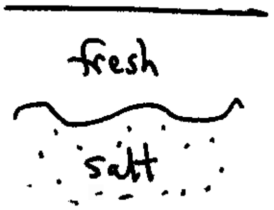
$$e^{-i\omega t}$$

corresponds to an exponentially growing mode, an *instability*. We see that, for $\nu = 0$, the interface is unstable whenever there is a discontinuity of the tangential velocity. This is called the *Kelvin-Helmholtz* instability. As Helmholtz remarked in 1868, 'every perfect geometrically sharp edge by which fluid flows must tear it asunder ... however slowly the rest of the fluid may move.'

Next, let $V_1 = V_2 = 0$, but now with $\rho_1 \neq \rho_2$ and both nonzero. We find

$$\omega^2 = \left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right) gk$$

For $\rho_1 > \rho_2$ (a heavy stratum below a light stratum), $\omega^2 > 0$ and so $\omega$ is real. Then the surface waves are simple oscillations. This equation can lead to interesting phenomena. For example, in an estuary, we might have a layer of fresh water above a layer of salt water. For this situation



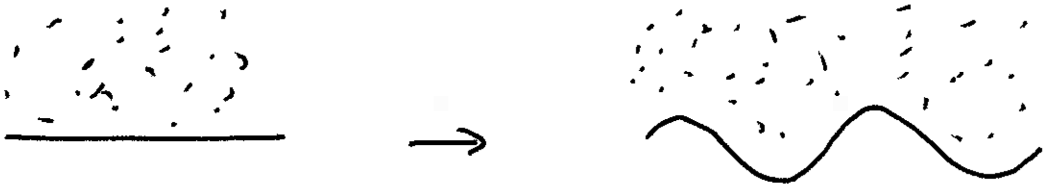
$$\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right) \sim 10^{-2}$$

Then the surface of the water can be perfectly smooth while a slow wave, with velocity about 10% of that of a surface wave, propagates at the interface.

On the other hand, if $\rho_1 < \rho_2$ (a heavy stratum above a light stratum)

$$\omega = \pm i \left[ \left( \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right) g k \right]^{\frac{1}{2}}$$

and the interface is unstable. Modes with larger $k$ grow with faster exponentials. Modes with very small wavelength or very large $k$ are stabilized by surface tension, an effect that we have not included in this analysis. So there is a value of $k$ that leads to the fastest instability. Then we would see fingering at wavelengths near this preferred one.



This is called the *Rayleigh-Taylor instability*

The general formula for the frequency of oscillations at the interface that comes from this analysis is

$$\omega = \left( \frac{\rho_1 V_1 + \rho_2 V_2}{\rho_1 + \rho_2} \right) k$$

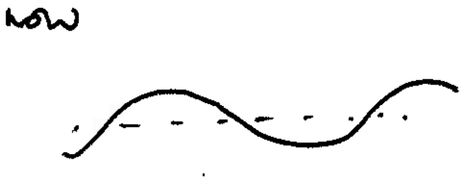
$$\pm \frac{1}{\rho_1 + \rho_2} \left[ (\rho_1^2 - \rho_2^2) g k - \rho_1 \rho_2 (V_1 - V_2)^2 k^2 \right]^{\frac{1}{2}}$$

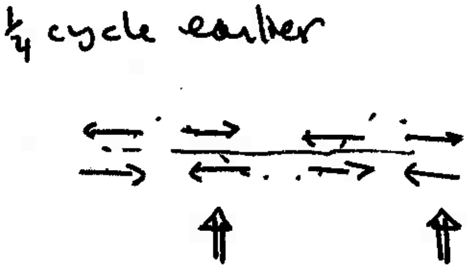
So, if $\rho_1 < \rho_2$, there is a competition between the stabilizing effect of gravity and the destabilizing effect of the Kelvin-Helmholtz shear layer. The interface is unstable if

$$(V_1 - V_2)^2 > \left( \frac{\rho_1^2 - \rho_2^2}{\rho_1 \rho_2} \right) \frac{g}{k}$$

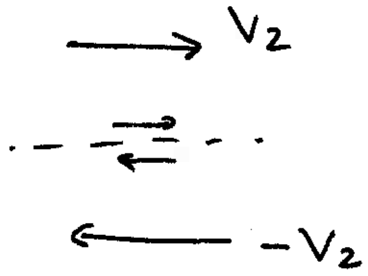
a condition always attained (ignoring surface tension) at sufficiently large $k$.

Batchelor gives the following intuitive explanation of the Kelvin-Helmholtz instability. Think, in particular, about the situation $V_1 = -V_2$, which is always realized in some frame. An oscillating surface at the interface requires the flows



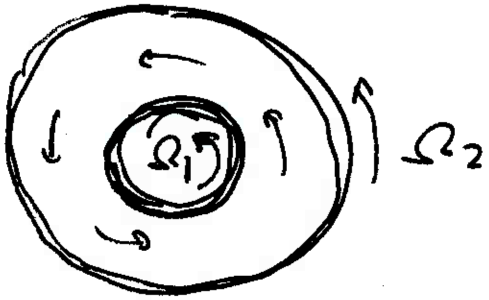


At the marked point, the oscillation leads to a vorticity with the same sense as the vorticity of the large-scale flow



There is constructive interference, and so we can expect the vorticity at this point to increase exponentially.

I would now like to consider an example with an instability in the bulk fluid. A simple solution to the Navier-Stokes equation is the laminar flow between rotating cylinders, called *Couette flow.*



In this geometry, I will take the inner radius to be $R_1$ and the outer radius to be $R_2$.

I will give a simplified analysis of the instabilities in this geometry by ignoring viscosity: $\nu = 0$. At the end of the discussion, I will sketch how the results change when viscosity is added back.

To analyze this problem, we perturb about a situation of steady flow. The Euler equation is

$$\vec{V}\cdot\vec{\nabla}\,\vec{V} = -\vec{\nabla}\,\frac{P}{\rho}$$

with the boundary condition $v_r(R_1) = v_r(R_2) = 0$. With no viscosity, the layers of fluid slip smoothly on one another, so any flow in the $\hat{\phi}$ direction that is uniform in $z$ solves the problem,

$$\vec{V} = V_\phi(r)\,\hat{\phi}$$

The Euler equation gives the corresponding pressure

$$\rho\,\frac{V_\phi^2}{r} = \frac{\partial}{\partial r}\,P$$

However, it might be best to perturb about the physical equilibrium solution with viscosity included. This situation is given by the solution above where, from the $\hat{\phi}$ component of the Navier-Stokes equation

$$\nu\,\nabla^2\vec{V} = 0$$

and the boundary conditions

$$\vec{v}(R_1) = \Omega_1 R_1 \hat{\phi}$$

$$\vec{v}(R_2) = \Omega_2 R_2 \hat{\phi}$$

are satisfied. The solution to these equations is

$$v_\phi(r) = ar + \frac{b}{r}$$

with

$$a = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_1^2 + R_2^2} \qquad b = \frac{(\Omega_2 - \Omega_1) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

The solution does not involve $\nu$, and now I will analyze its stability setting $\nu = 0$.

I will first present an intuitive argument for the stability boundary. Consider the general inviscid case, for which any function

$$v_\phi(r)$$

gives a solution. Because there is no friction, the layers cannot transfer angular momentum from one to the other. A perturbation might cause a fluid layer to move to a larger or smaller radius; this is the only way that angular momentum can be moved around.

Let $H = rv_\phi$. Then the angular momentum density is

$$\rho r v_\phi = \rho H$$

For the zeroth-order flow in the $\hat{\phi}$ direction, the kinetic energy density is

$$\frac{1}{2}\rho v_\phi^2 = \frac{1}{2}\rho \frac{H^2}{r^2}$$

Now imagine that the fluid is perturbed in such a way that the layers of fluid at radii $r_1 < r_2$ are interchanged. The kinetic energy before the motion is

$$\frac{1}{2}\rho\left(\frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2}\right)$$

The kinetic energy after the motion is

$$\frac{1}{2}\rho\left(\frac{H_1^2}{r_2^2} + \frac{H_2^2}{r_1^2}\right)$$

The change in energy is

$$\Delta(KE) = \frac{1}{2}\rho \left(H_2^2 - H_1^2\right)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

Now, $r_1 < r_2$, so the second factor is positive. Thus, if $H_2^2 < H_1^2$, there is a motion of the fluid that lowers the energy. The flow will be unstable to this deformation. This gives the follow criterion for stability:

$$\frac{d}{dr} H(r) > 0$$

An equivalent condition is

$$\Phi(r) > 0 \qquad \Phi = \frac{1}{r^3}\frac{d}{dr}\left(rV(r)\right)^2$$

The quantity $\Phi$ is called the *Rayleigh discriminant.*

Return now to the particular solution discussed above, arising from viscous Couette flow. Write the general form of the flow as a combination of the zeroth-order solution and small perturbations around it.

$$\vec{v} = V(r)\,\hat{\phi} + \delta v_r\,\hat{r} + \delta v_\phi\,\hat{\phi} + \delta v_z\,\hat{z}$$

$$p = P(r) + \delta p$$

At an appropriate point, I will replace $\delta v_r$, $\delta v_\phi$, etc. with Fourier components with coefficients called $v_r$, $v_\phi$, ... in accord with the convention discussed above.

To begin, write the three components of the Euler equation in cylindrical coordinates

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial}{\partial r} v_r + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} v_r + v_z \frac{\partial}{\partial z} v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial}{\partial r} p$$

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial}{\partial r} v_\phi + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} v_\phi + v_z \frac{\partial}{\partial z} v_\phi + \frac{v_\phi v_r}{r} = -\frac{1}{\rho r}\frac{\partial}{\partial \phi} p$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial}{\partial r} v_z + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} v_z + v_z \frac{\partial}{\partial z} v_z = -\frac{1}{\rho}\frac{\partial}{\partial z} p$$

The equation of continuity takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \phi} v_\phi + \frac{\partial}{\partial z} v_z = 0$$

Now insert the expansion about steady Couette flow, keeping terms to first order in small quantities

$$\frac{\partial}{\partial t}\delta v_r + \frac{V}{r}\frac{\partial}{\partial \phi}\delta v_r - \frac{V^2 + 2V\delta v_\phi}{r} = -\frac{1}{\rho}\left(\frac{\partial}{\partial r}P + \frac{\partial}{\partial r}\delta p\right)$$

$$\frac{\partial}{\partial t}\delta v_\phi + \delta v_r \frac{\partial}{\partial r}V + \frac{V}{r}\frac{\partial}{\partial \phi}\delta v_\phi + \frac{V}{r}\delta v_r = -\frac{1}{\rho r}\frac{\partial}{\partial \phi}\delta p$$

$$\frac{\partial}{\partial t}\delta v_z + \frac{V}{r}\frac{\partial}{\partial \phi}\delta v_z = -\frac{1}{\rho}\frac{\partial}{\partial z}\delta p$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\delta v_r) + \frac{1}{r}\frac{\partial}{\partial \phi}\delta v_\phi + \frac{\partial}{\partial z}\delta v_z = 0$$

Now introduce Fourier components. The problem is uniform in $t$, $z$, and $\phi$, so we can Fourier decompose in all three variables

$$\delta v_r = v_r(r)\; e^{-i\omega t} e^{ikz} e^{in\phi}$$

$$\delta v_\phi = v_\phi(r)\; e^{-i\omega t} e^{ikz} e^{in\phi}$$

etc.

Note that the system is periodic in $\phi$, so $n$ must be an integer.

From here on, I will specialize to *axisymmetric* perturbations, $n = 0$. It can be shown that these are the leading instabilities. The above equations becomes

$$
\begin{aligned}
-i\omega v_r - 2\Omega v_\phi &= -\frac{1}{\rho} D P \\
-i\omega v_\phi + (D_* V)\, v_r &= 0 \\
-i\omega v_z &= -ik \frac{1}{\rho} P \\
D_* v_r + ik v_z &= 0
\end{aligned}
$$

with

$$
\Omega = \frac{V(r)}{r} \qquad D = \frac{d}{dr} \qquad D_* = \frac{d}{dr} + \frac{1}{r} = \frac{1}{r}\frac{d}{dr} r
$$

It is easy to solve the second, third, and fourth equations

$$
v_\phi = -i \frac{1}{\omega} (D_* V)\, v_r
$$

$$
\frac{P}{\rho} = \frac{\omega}{k} v_z = i \frac{\omega}{k^2} D_* v_r
$$

Plugging the result back into the first equation, we find

$$
-i\omega v_r + i \frac{1}{\omega} 2\Omega (D_* V)\, v_r = -i \frac{\omega}{k^2} D D_* v_r
$$

The Rayleigh discriminant defined above becomes, for this situation,

$$\Phi = \frac{1}{r^3}\frac{d}{dr}(rV)^2 = 2\,\frac{1}{r^2}\,rV D_* V = 2\Omega D_* V$$

Thus, the equation for $v_r$ takes the form

$$\left(-DD_* + k^2\right) V_r = \frac{k^2}{\omega^2}\,\Phi\, V_r$$

This is an ordinary differential equation that must be solved for $v_r(r)$ with the boundary conditions

$$V_r(R_1) = V_r(R_2) = 0$$

The equation we have found is similar to a Schrödinger equation with a potential that depends on $\Phi(r)$. In this language, we want to find a zero-energy eigenfunction of this equation. The condition that the energy is zero gives us $\omega$ in terms of $k$.

Before looking at explicit examples of solutions, it is worth doing some general analysis. Multiply by $v_r^*$ and integrate $\int r dr$. This gives

$$0 = \int_{R_1}^{R_2} dr\; r\; V_r^* \left(-\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r + k^2 - \frac{k^2}{\omega^2}\Phi(r)\right) V_r$$

In the first term, integrate by parts. The surface term vanishes by the boundary conditions. We find

$$\int_{R_1}^{R_2} dr \left( \frac{1}{r} \left| \frac{d}{dr} r v_r \right|^2 + r k^2 |v_r|^2 \right) = \frac{k^2}{\omega^2} \int_{R_1}^{R_2} dr\, r\, \Phi(r) |v_r|^2$$

The left-hand side is positive. Essentially, we are using the fact that $D$ and $D_*$ are adjoint operators under this inner product, that is,

$$\int_{R_1}^{R_2} dr\, r\, f D g = \int_{R_1}^{R_2} dr\, r\, f \frac{d}{dr} g = -\int_{R_1}^{R_2} dr\, r \left( \frac{1}{r} \frac{\partial}{\partial r} r f \right) g$$

and, therefore, that

$$= \int_{R_1}^{R_2} dr\, r [(-D_*) f]\, g$$

$$\Delta = -D D_*$$

is a self-adjoint operator.

From the final result, we see that, if $\Phi(r) > 0$, the only solutions have $k^2/\omega^2 > 0$ or $\omega^2 > 0$ for real $k$. These are stable oscillations about the zeroth-order solution. On the other hand, if $\Phi(r) < 0$ in some region, we can find an eigenfunction $v_r(r)$ that takes advantage of this to generate a solution with $\omega^2 < 0$. That would signal an instability.

Now put in the explicit formulae for Couette flow. First

$$\Phi = \frac{1}{r^3} \frac{d}{dr} (rV)^2 = \frac{1}{r^3} \frac{d}{dr} (ar^2 + b)^2 = 4a \left( a + \frac{b}{r^2} \right)$$

that is

$$\Phi = 4a\,\Omega(r) \qquad \Omega(r) = \frac{V(r)}{r} = a + \frac{b}{r^2}$$

where

$$a = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$$

I have set up the problem with $R_2 > R_1$ and $\Omega_2 > 0$. There are still two cases, $\Omega_1 > 0$ and $\Omega_1 < 0$.

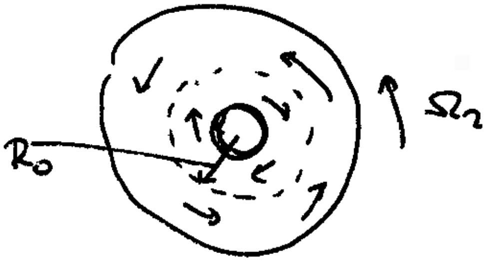
Consider first $\Omega_1 > 0$. In this case, $\Omega(r) > 0$. Then

$$\Phi = 4\left(\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\right)\Omega$$

This is positive only if $\Omega_2 R_2^2 - \Omega_1 R_1^2 > 0$, so the condition for stability is

$$\frac{\Omega_2}{\Omega_1} > \frac{R_1^2}{R_2^2}$$

If $\Omega_1 < 0$, $a > 0$, but $\Omega(r)$ goes through zero at some radius $R_0$ between $R_1$ and $R_2$.



In the region $r < R_0$, $\Phi < 0$, and so we can expect to find instabilities based in this region.

To make this concrete, we can simplify to the *narrow gap* approximation, $\ell = R_2 - R_1 \ll R_2$. Write

$$r = R_1 + \ell x \qquad 0 < x < 1$$

Then

$$\Omega = \Omega_1 + (\Omega_2 - \Omega_1)x + O(x^2)$$

to first order in $x$. The boundary condition on $v_r(r)$ is

$$v_r = 0 \quad \text{at} \quad x = 0 \;, \; x = 1$$

The differential operators become

$$D = \frac{d}{dr} = \frac{1}{\ell}\frac{d}{dx}$$

$$D_* = \frac{d}{dr} + \frac{1}{r} = \frac{1}{\ell}\frac{d}{dx} + O(\ell^0) \approx \frac{1}{\ell}\frac{d}{dx}$$

and $\Phi$ takes the form

$$R \approx R_1 \approx R_2$$

$$\Phi = \frac{1}{r^3} \frac{d}{dr}\left(r^2\Omega\right)^2 = \frac{2r}{\ell} \frac{d\Omega}{dx} \Omega + \cdots = \frac{2R}{\ell} (\Omega_2 - \Omega_1)\, \Omega(x)$$

The problem with $\Omega_1 = \Omega_2 = \Omega$, a bulk rotation of the fluid, is simple to analyze and thus a good place to start, even though we do not expect any interesting behavior in this limit. The value of $\Phi$ is

$$\Phi = 4\Omega^2$$

The equation for $v_r$ becomes the eigevalue problem

$$\left( -\frac{1}{\ell^2} \frac{d^2}{dx^2} + k^2 \right) v_r = \frac{k^2}{\omega^2}\, 4\Omega^2\, v_r$$

Since $v_r$ must vanish at $x = 0, 1$, the eigenfunctions of the operator on left-hand side are

$$v_r = \sin(m\pi x) \qquad m = \text{integer}$$

Then

$$(m\pi)^2 + (k\ell)^2 = \frac{(k\ell)^2}{\omega^2} \cdot 4\Omega^2$$

and so the frequencies of oscillation about the unperturbed rotation are

$$\omega_m^2 = \frac{(k\ell)^2}{(k\ell)^2+(m\pi)^2}\cdot 4\Omega^2$$

All of these frequencies satisfy $\omega_m^2 > 0$, so we have only stable oscillations, as expected.

Now consider the caes of $\Omega_1$ negative. Let $\Delta\Omega = \Omega_2 - \Omega_1$. The eigenvalue equation is now

$$\left(-\frac{1}{\ell^2}\frac{d^2}{dx^2}+k^2\right)v_r = \frac{k^2}{\omega^2}\,\frac{2R}{\ell}\,\Delta\Omega\,(\Delta\Omega x - |\Omega_1|)\,v_r$$

or

$$\left(-\frac{d^2}{dx^2}+(k\ell)^2-\frac{(k\ell)^2}{\omega^2}\,\frac{2R}{\ell}\left[(\Delta\Omega)^2 x - \Delta\Omega|\Omega_1|\right]\right)v_r = 0$$

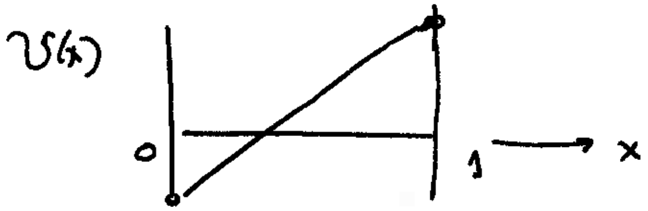
This is a Schrödinger equation on the interval $(0,1)$ with a linear potential. An instability occurs if there is a solution with $\omega^2 < 0$. This is a solution to

$$\left[-\frac{d^2}{dx^2}+\mathcal{V}(x)\right]v_r = 0$$

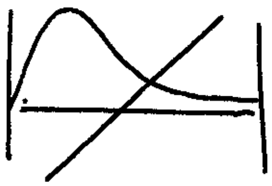
with

$$\mathcal{V}(x) = (k\ell)^2+\frac{(k\ell)^2}{|\omega^2|}\,\frac{2R}{\ell}\left((\Delta\Omega)^2 x - \Delta\Omega|\Omega_1|\right)$$

For small enough $|\omega^2|$, this potential has the form



and so for small $|\omega^2|$ we can find a zero-energy eigenfunction of the form



For even smaller $|\omega^2|$, there are further solutions, but the solution with the largest imaginary part of $\omega$ is the strongest instability. This example shows explicitly how the flow is able to take advantage of a negative value of $\Phi(r)$ in some region to find a way to go unstable.

The study of the instabilities of Couette flow with $\nu \neq 0$ is considerably more complicated. The analysis of the instability and its experimental verification was presented by G. I. Taylor in a landmark work. The eigenvalue equation in this case takes the more complicated form

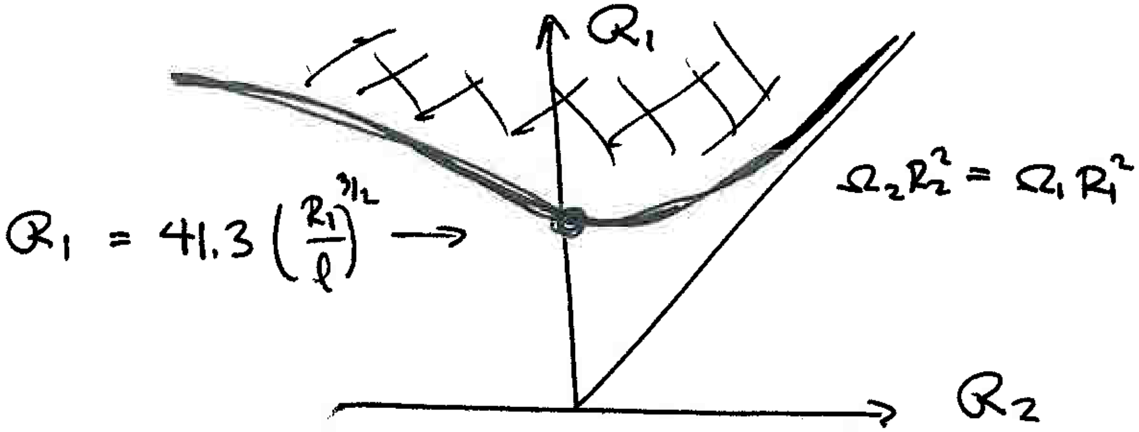
$$[\nu(DD_* - k^2) + i\omega][DD_* - k^2]v_r = 2k^2\Omega v_\phi$$

$$(\nu DD_* - k^2)v_\phi = D_*V v_r$$

Intuitively, the nonzero viscosity should quench the instability for low Reynolds number, and this is indeed the case. Define two Reynolds numbers

$$\mathcal{R}_1 = \frac{\Omega_1 R_1^2}{\nu} \qquad \mathcal{R}_2 = \frac{\Omega_2 R_2^2}{\nu}$$

Then, the stability boundary in the $\mathcal{R}_1, \mathcal{R}_2$ plane has the form



where Couette flow is instable in the shaded region. The situation for $\mathcal{R}_1 \to \infty$ agrees with the Rayleigh criterion and the results derived above.