Jan. 28

Turbulence

In this lecture, I will discuss some aspects of the very complex flows found at high Reynolds number. Flows of this type are called *fully developed turbulence.* To prepare for this study, I will first discuss some further aspects of the convection of vorticity.

We saw in our discussion of boundary layers, we analyzed vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{v}$$

in 2-dimensional flows. In such flows, vorticity points perpendicular to the plane of the flow and so can be treated as a scalar. It obeys the diffusion equation

$$\frac{D\vec{\omega}}{Dt} = \frac{\partial \vec{\omega}}{\partial t} + \vec{v}\cdot\vec{\nabla}\,\vec{\omega} = \nu \nabla^2 \vec{\omega}$$

In particular, in the limit of small viscosity, vorticity is conserved along a streamline.

In 3 dimensions, the situation is more complicated. Look again at the derivation of the above equation. We wrote

$$\vec{v} \times \vec{\omega} = \vec{v} \times (\vec{\nabla} \times \vec{v}) = v^i\, \vec{\nabla}\, v^i - \vec{v}\cdot\vec{\nabla}\,\vec{v}$$

so that

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{v} \times \vec{\omega} + \frac{1}{2}\vec{\nabla} v^2$$

Now take the curl of the Navier Stokes equation. The curl of a gradient is zero. The second term simplifies by

$$\vec\nabla \times [\vec v\cdot\vec\nabla\, \vec v] = \vec\nabla\times(-\vec v\times\vec\omega) = -\nabla^i(\vec v\, \omega^i) + \nabla^i(v^i\,\vec\omega)$$

$$= -\vec\omega\cdot\vec\nabla\, \vec v - \vec v\,(\vec\nabla\cdot\vec\omega) + (\vec\nabla\cdot\vec v)\,\vec\omega + (\vec v\cdot\vec\nabla)\,\vec\omega$$

Using $\vec\nabla\cdot\vec\omega = 0 = \vec\nabla\cdot\vec v$, we find

$$\vec\nabla\times\left(\frac{\partial\vec v}{\partial t} + \vec v\cdot\vec\nabla\,\vec v = -\vec\nabla\frac{p}{\rho} + \nu\nabla^2\vec v\right)$$

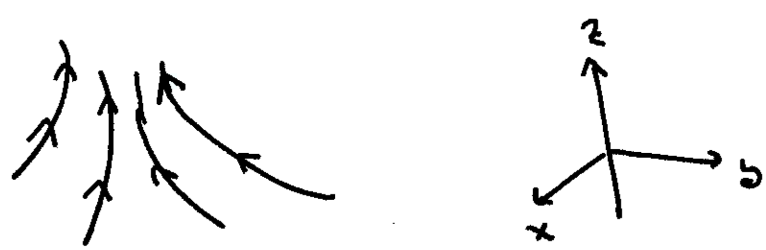
becoms

$$\frac{\partial}{\partial t}\vec\omega + \vec v\cdot\vec\nabla\,\vec\omega = (\vec\omega\cdot\vec\nabla)\,\vec v + \nu\nabla^2\vec\omega$$

or

$$\frac{D}{Dt}\vec\omega = (\vec\omega\cdot\vec\nabla)\,\vec v + \nu\nabla^2\vec\omega$$

This equation differs from the convective diffusion equation by the first term on the right. To see the effect of this term, imagine a region with a converging flow



$$\frac{\partial v_x}{\partial x} < 0 \qquad \frac{\partial v_y}{\partial y} < 0 \qquad \frac{\partial v_z}{\partial z} > 0$$

$$\text{so} \quad \vec{\nabla}\cdot\vec{v} = 0$$

Add a vorticity

$$\vec{\omega} \parallel \hat{z}$$

Then

$$\frac{D}{Dt}\vec{\omega} \sim \frac{\partial v_z}{\partial z}\vec{\omega}$$

and the vorticity grows exponential. Stretching a flow then spins up the vorticity.

Here is a slightly more formal argument to the same point. Multiply the above equation by $\vec{\omega}$. Then

$$\frac{D}{Dt}\omega^2 = \omega^i(\omega^j\nabla^j)v^i + (\text{term} \propto \nu)$$

If $\vec{\nabla}\cdot\vec{v} = 0$,

$$\tfrac{1}{2}(\nabla^j v^l + \nabla^l v^j) = \sigma^{lj}$$

and so

$$\frac{D}{Dt}\omega^2 = \omega^i \omega^j \sigma^{ij}$$

The shear σ^{ij} is a traceless symmetric tensor, so, if we diagonalize σ, the three eigenvalues λ_i must satisfy

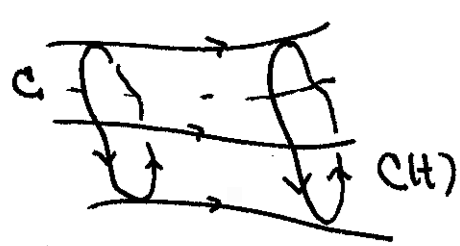
$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

Thus, at least one of the λ_i must be positive. Assume $\lambda_1 > 0$. Then if ω_1 is the component of $\vec{\omega}$ along the principal axis 1,

$$\frac{D}{Dt}(\omega_1^2 + \cdots) = \lambda_1 \omega_1^2$$

up to effects of viscosity. Thus, any flow at high Reynolds number *amplifies* vorticity along some axis.

Let me give one more argument for this effect, making more direct use of the Navier-Stokes equation. Let C be a closed curve in the fluid. As the fluid moves, the curve is convected along



Let $C(t)$ be the position of the curve at time t. Define the *circulation*

$$\Gamma = \oint_C d\vec{\ell} \cdot \vec{v}$$

and, similarly, let $\Gamma(t)$ be the circulation around $C(t)$ at time t. *Kelvin's circulation theorem* states that, for $\nu = 0$,

$$\frac{d}{dt}\Gamma(t) = 0$$

Here is the proof: To compute $d\Gamma/dt$, we add the change in $\vec{v}$ on the curve from the explicit time-dependence of the flow to the change in $\vec{v}$ on the curve from the motion of the curve in the fluid.

$$\frac{d\Gamma}{dt} = \oint_C d\ell \cdot \frac{\partial \vec{v}}{\partial t} + \oint_C d\vec{\ell}\, (\vec{v} \cdot \vec{\nabla})\vec{v}$$

Then

$$\frac{d\Gamma}{dt} = \oint d\vec{\ell} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right]$$

$$= \oint d\vec{\ell} \cdot \left(-\frac{\vec{\nabla} p}{\rho} \right) = 0$$

because the pressure is a single-valued scalar function.

A consequence of Kelvin's theorem is that, if the flow is such as to make the loop smaller, the velocity of a flow around the loop, and also $\vec{\omega} = \vec{\nabla} \times \vec{v}$, must increase to compensate.

These ideas imply that a large-scale flow can speed up flows on smaller scales. Applying this idea on successively smaller scales, we can see how a fluid flow can become increasingly complex. The large-scale flows stretch and bend the geometry of the flow. This spins up vortices at the next smaller scale. THe flows in these vortices stretch and bend the fluid flow and affect the next level of sizes. If the process continues down many levels, the flow becomes very complex indeed.

The smaller-scale flows also feed back on the large-scale flows and draw out their energy. We can see in a rough way how this works by rearranging the Navier-Stokes equation. Divide the fluid velocity into two parts

$$\vec{v} = \vec{v}_{av} + \delta\vec{v}_{sm}$$

at a size scale ℓ. Averaging over regions of size ℓ gives $\vec{v}_{av}$. The residual motions that are not included in $\vec{v}_{av}$ are put in the piece associated with small length scales $\delta\vec{v}_{sm}$. Then, the average of $\vec{v}_{sm}$ over a region of size ℓ gives

$$\langle \delta\vec{v}_{sm} \rangle = 0 \quad \underline{\text{but}} \quad \langle \delta v^i_{sm}\, \delta v^j_{sm} \rangle \neq 0$$

This will be important in our analysis.

Now average the Navier-Stokes equation over regions of size ℓ.

$$\rho \frac{\partial}{\partial t} \vec{v}_{av} + \rho \langle \vec{v} \cdot \vec{\nabla}\, \vec{v} \rangle = -\vec{\nabla} p_{av.} + \rho \nu \nabla^2 \vec{v}_{av}$$

We can break up

$$\langle (\vec{v} \cdot \vec{\nabla}) \vec{v} \rangle = (\vec{v}_{av} \cdot \vec{\nabla}) \vec{v}_{av} + \nabla^i \langle \delta v^i_{sm}\, \delta\vec{v}_{sm} \rangle$$

$$(\text{with} \quad \vec{\nabla} \cdot \delta\vec{v}_{sm} = 0\ )$$

We can put this last term on the right-hand side of the equation and call it the *Reynolds stress*

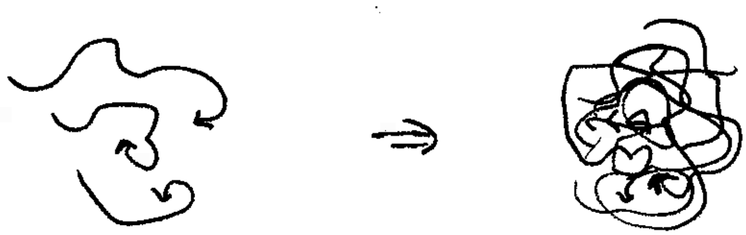
$$T_R^{ij} = \rho \langle \delta v_{sm}^i \, \delta v_{sm}^j \rangle$$

$$\rho \frac{\partial \vec{v}_{av}^{\,j}}{\partial t} + \vec{v}_{av} \cdot \vec{\nabla} \, \vec{v}_{av}^{\,j} = - \vec{\nabla}^j p_{av} + \eta \nabla^2 \vec{v}_{av}^{\,j} - \nabla^i T_R^{ij}$$

T_R^{ij} depends on the large-scale flow that is driving the small-scale velocities. We can parametrize this dependence by expanding T_R in the large-scale velocity and its derivatives, as we do in deriving constituitive relations. Thus, write

$$T_R^{ij} = P_t \, \delta^{ij} - 2 \rho \nu_t \, \sigma_{av}^{ij} + \cdots$$

where σ_{av}^{ij} is the stress tensor of the large-scale flow, and the omitted terms have higher derivatives of $\vec{v}_{av}$. This representation is sensible if the small-scale flows are approximately isotropic. This may not be the case in a simple flow pattern, but we might hope that it would become a good approximation at very small scales if the flows become very complex



Then

$$\rho\left[\frac{\partial V_{av}}{\partial t} + \vec{V}_{av}\cdot\vec{\nabla}\,\vec{V}_{av}\right]$$

$$= -\vec{\nabla}p_{av} + \rho\nu\nabla^2\vec{V}_{av} - \vec{\nabla}\mathcal{P}_t + \rho\nu_t\nabla^2\vec{V}_{av} + \cdots$$

We can estimate $\mathcal{P}_t$, ν_t by dimensional analysis, using the velocity and size scale of the smallest flows,

$$\mathcal{P}_t \sim \rho(\delta v_{sm})^2 \qquad \text{"turbulent pressure"}$$

$$\nu_t \sim \delta v_{sm}\,\ell \qquad \text{"turbulent viscosity"}$$

Notice that

$$\frac{\nu_t}{\nu} = \frac{\delta v_{sm}\,\ell}{\nu}$$

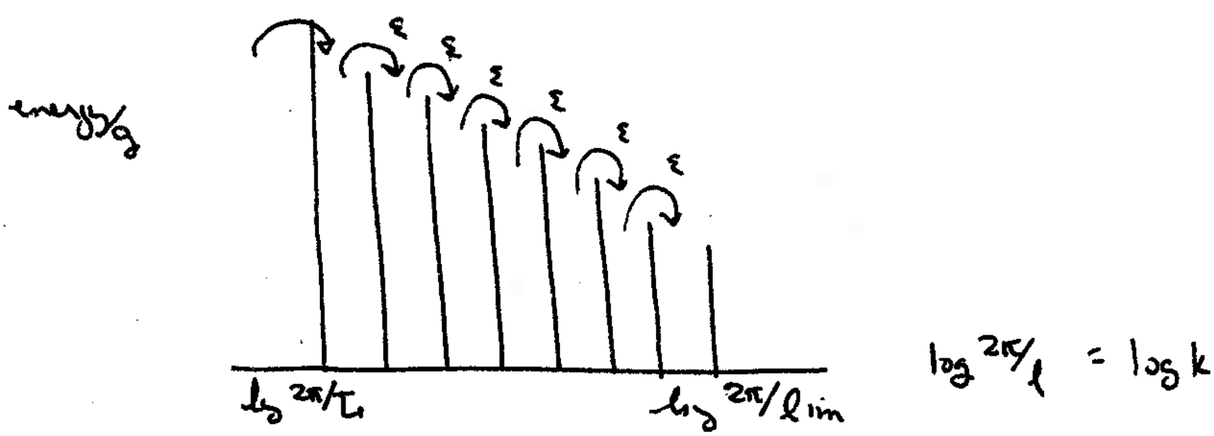
The quantity on the right is the Reynolds number at the smallest scale. So, typically $\nu_t/\nu \gg 1$ for a complex flow. *Much* more energy is sucked out of the large-scale flow by ints interaction with smaller-scale vortices than by the original viscosity term in the Navier-Stokes equation.

Kolmogorov took this idea and built up a theory of fully developed turbulence by looking at energy flow from one scale to another in a statistical way. The ideas of Kolmogorov's picture are the following:

1. The primary quantity is the energy dissipation per unit mass ϵ. The units of ϵ are

$$\frac{\dot{\rho}_\epsilon}{\rho} \sim \frac{\text{g cm}^2/\text{sec}^2 \;/\; \text{cm}^3\,\text{sec}}{\text{g}/\text{cm}^3} \sim \text{cm}^2/\text{sec}^3$$

2. Fluid motions interact *locally* in Fourier space or *locally in scale.* This assumption implies that the dissipated energy is passed down in scale from the large-scale motions to smaller eddies. Equivalently, the energy is passed up in wavenumber k from $k = 2\pi/L$, where L is the size of the large eddies, to $k = 2\pi/\ell_{min}$, where ℓ_{min} is the size of the smallest eddies. In fully developed turbulence $L \gg \ell_{min}$, so this process requires many stages.



3. Until we reach the bottom of the cascade, ν is irrelevant. Then we can estimate dynamical quantities by dimensional analysis, ignoring ν.

These assumptions have the flavor of those used in the *Landau mean field theory* of second-order phase transitions. In the statistical-mechanics context, that theory is known to be qualitatively correct but not exact, even for the scaling laws. In the theory of turbulence, the true or exact scaling theory has not yet been discovered.

We can now work out the consequences of Kolmogorov's assumptions. At a scale ℓ such that $L \gg \ell \gg \ell_{min}$, the eddies have a characteristic velocity v_ℓ. The turbulent viscosity at this scale is then

$$\nu_t \sim v_\ell \cdot \ell$$

Turbulent viscosity does not *destroy* energy (or, better, convert it to heat). Instead, it *transfers* energy from each layer in ℓ or k to the next layer at smaller length scales. The rate of energy transfer is

$$\varepsilon \approx \nu_t \left(\frac{\partial V_{sm}}{\partial x}\right)^2 \sim \frac{V_\ell^3}{\ell} \qquad cm^2/sec^3$$

We keep ϵ fixed as ℓ varies, so

$$V_\ell \sim (\varepsilon \ell)^{1/3}$$

If L is the original scale of the problem and V_0 is the fluid velocity in the driving flow,

$$V_\ell \sim V_0 \left(\frac{\ell}{L}\right)^{1/3}$$

Now we can address the length of the cascade. As ℓ decreases, v_ℓ also decreases, and so ν_t decreases. At some stage, ν_t becomes sufficiently small that the original fluid viscosity ν can become relevant. Then we will have true dissipation, conversion of mechanical energy to heat. At this point, the cascade ends. This situation occurs when

$$1 \sim \frac{\nu_t}{\nu} \sim \frac{V_\ell \ell}{\nu} \sim \frac{\varepsilon^{1/3} \ell^{4/3}}{\nu}$$

$$\sim \frac{V_0}{\nu} \frac{\ell^{4/3}}{L^{1/3}} \sim \frac{V_0 L}{\nu} \left(\frac{\ell}{L}\right)^{4/3}$$

We recognize the large-scale Reynolds number $R = V_0 L/\nu$. The smallest scale of the cascade is

$$\ell_{min} \sim L / R^{3/4}$$

The most famous consequence of the Kolmogorov theory is its consequence for the energy spectrum. At the scale ℓ, the contribution to the energy density is

$$(\rho\epsilon)_\ell = \frac{1}{2}\rho v_\ell^2 \sim \rho\,\epsilon^{2/3}\,\ell^{2/3}$$

or

$$\frac{(\rho\epsilon)_\ell}{\rho} \sim \frac{\varepsilon^{2/3}}{k^{2/3}} \qquad k = \frac{2\pi}{\ell}$$

Sum this over scales, with an integral

$$\int \frac{dk}{k}$$

to preserve the dimension

$$\frac{\rho\epsilon}{\rho} \sim \frac{g\,cm^2/sec^2\,/cm^3}{g/cm^3} \sim cm^2/sec^2$$

Then the total energy per unit mass is

$$\frac{\rho_\epsilon}{\rho} = \int_{2\pi/L}^{2\pi/\ell_{min}} dk \quad \mathcal{U}(k)$$

with

$$\mathcal{U}(k) \sim \frac{\varepsilon^{2/3}}{k^{5/3}}$$

This result for the power spectrum of fully-developed turbulence is the *Kolmogorov 5/3-law.* It is a reasonably good description of experimental data spanning up to 5 orders of magnitude in length scale.

Kolmogorov theory also makes predictions for correlation functions of velocity in the flow. The simplest of these is

$$\left\langle \left| \vec{V}(x+r) - \vec{V}(x) \right|^2 \right\rangle \sim (\varepsilon r)^{2/3} \qquad \mathrm{cm}^2/\mathrm{sec}^2$$

The derivation is again by dimensional analysis. If $r \gg \ell_{min}$, ϵ and r are then only available dimensional quantities. We can get a little more information by looking at the tensor structure of this quantity. Write

$$D^{ij}(\vec{r}) = \left\langle [V^i(\vec{x}+\vec{r}) - V^i(\vec{x})][V^j(\vec{x}+\vec{r}) - V^j(\vec{x})] \right\rangle$$

For homogeneous, isotropic turbulence, the tensor decomposition is

$$D^{ij}(\vec{r}) = A(r)\,\delta^{ij} + B(r)\,\hat{r}^i\hat{r}^j$$

Let

$$\vec{x}_1 = \vec{x}+\vec{r} \qquad x_2 = \vec{x}$$

and use the notation

$$\vec{V}_1 = \vec{V}(\overrightarrow{x+r}) \qquad \vec{V}_2 = \vec{V}(\vec{x})$$

Then

$$D^{ij} = \langle v_1^i v_1^j \rangle - \langle v_1^i v_2^j \rangle - \langle v_2^i v_1^j \rangle + \langle v_2^i v_2^j \rangle$$

and, by the assumption of homogeneity and isotropy

$$D^{ij} = 2\left(\langle v_1^i v_1^j \rangle - \langle v_1^i v_2^j \rangle\right)$$

Finally, apply $\vec{\nabla} \cdot \vec{v} = 0$, which implies

$$\frac{\partial}{\partial x_2^j} D^{ij} = 2 \frac{\partial}{\partial x_2^j} \left(\langle v_1^i v_1^j \rangle - \langle v_1^i v_2^j \rangle \right)$$

$$= 2 \cdot \left(0 - \langle v_1^i \vec{\nabla} \cdot \vec{v}_2 \rangle \right) = 0$$

so that

$$\frac{\partial}{\partial \vec{r}^{\,j}} D^{ij}(\vec{r}) = 0$$

In terms of A and B,

$$0 = \hat{r}^i \frac{\partial A}{\partial r} + \hat{r}^i \frac{\partial B}{\partial r} + B \, \nabla^j \left(\frac{r^i r^j}{r^2} \right)$$

$$= \hat{r}^i \frac{\partial}{\partial r}(A+B) + B \, (3+1-2) \frac{r^i}{r^2}$$

$$= \hat{r}^i \left(\frac{\partial A}{\partial r} + \frac{\partial B}{\partial r} + 2\frac{B}{r} \right) \quad \underline{\text{so}} \quad \frac{\partial A}{\partial r} = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 B$$

Both A and B are proportional to $r^{2/3}$, so

$$\frac{2}{3} A = -\frac{8}{3} B \quad \text{or} \quad B = -\frac{1}{4} A$$

This relation is usually expressed in terms of velocity correlations *parallel* and *perpendicular* to the line of separation

$$D_{LL} = \langle \hat{r} \cdot (\vec{v}_1 - \vec{v}_2) \; \hat{r} \cdot (\vec{v}_1 - \vec{v}_2) \rangle$$

$$D_{NN} = \langle \hat{n} \cdot (\vec{v}_1 - \vec{v}_2) \; \hat{n} \cdot (\vec{v}_1 - \vec{v}_2) \rangle$$

with

$$\hat{n} \cdot \hat{r} = 0 \qquad \text{Note that } D_{LN} = 0$$

by isotropy. You can see that

$$D_{LL} = A + B \qquad D_{NN} = A$$

It follows that

$$D_{LL} = C\, (\varepsilon r)^{2/3}$$

$$D_{NN} = \frac{4}{3}\, C\, (\varepsilon r)^{2/3}$$

where both the exponent and the ratio of the coefficients are predicted by the Kolmogorov theory.

In the 1970's, the best evidence for the Kolmogorov 5/3-law came from measurements of natural highly turbulent flows in Puget Sound. Very recently, it has become possible to observe turbulent flows in great detail under controlled conditions by using modern methods of imaging. As especially impressive recent reference is Voth, La Porta, Crawford, Alexander, and Bodenschatz, J. Fluid Mech. 469, 121 (2002). These authors generated turbulence in a large chamber, allowing a fiducial region well away from the walls of size 30 cm. They illuminated the turbulent fluid with lasers, placed small reference particles in the flow, and measured the positions of these particles using a silicon strip detector recycled from the CLEO-III elementary particle physics experiment. This allowed 1 μ accuracy in particle position measurements, at a frame rate of 70,000 frames per second. The scaling laws just discussed, and similar predictions of the Kolmogorov theory, were studied at values of the Reynolds number up to $R \sim 1000$. Though the error on the exponent is not quoted, the relations just above are in excellent agreement with the data.