Feb. 16

Heat Conduction and Convection

Up to this point in the course, we have been discussing structureless fluids with no internal degrees of freedom. These are fluids with constant density and no internal energy, obeying the incompressible Navier-Stokes equation. The behavior of fluids under these approximations is already complex enough to occupy us for a quarter (or even a lifetime), but now we will move on to consider fluids with various types of additional structure.

We have made reference to fluids that transport tracer species. The concentration of such a species obeys the convective diffusion equation

$$\frac{D}{Dt} c = \frac{\partial}{\partial t} c + (\vec{v}\cdot\vec{\nabla}) c = D_c \nabla^2 c$$

where D_c is the diffusion constant for the tracer. In this lecture, I will derive the generalization of this equation for the *internal energy* of a fluid and work out some applications of that equation.

We have written equations for the energy flow in a fluid twice before in this course. The equation for local energy conservation takes the form

$$\frac{\partial}{\partial t} \rho_\epsilon + \vec{\nabla}\cdot\vec{j}_\epsilon = 0$$

where ρ_ϵ is the energy density and $\vec{j}_\epsilon$ is the energy current or flux, the energy flow through a surface per cm^2 per second. The right-hand side of this equation should really be zero: Energy is conserved.

In the first lecture, I wrote an explicit form of this equation valid for an ideal fluid

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}v^2+u\right)\right] + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\tfrac{1}{2}v^2+h\right)\right] = 0$$

where u is the internal energy per gram and h is the enthalpy per gram of fluid.

$$u = \frac{E}{\rho V} \qquad h = \frac{H}{\rho V}$$

where

$$H = E + pV$$

When we added *viscosity*, we modified this equation in two ways. First, we modified the energy flux to include the work done by viscous forces

$$j^i_\epsilon = (\text{previous}) + T^{ij}_{visc} v^j$$

$$T^{ij}_{visc} = -2\eta\sigma^{ij} - \zeta\Theta\delta^{ij}$$

Second, we included the effect of the viscosity in dissipating mechanical energy into heat as a negative term on the right-hand side of the equation,

$$\frac{\partial}{\partial t}\rho_\epsilon + \vec{\nabla}\cdot\vec{j}_\epsilon = -\frac{1}{2}\eta\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3}\delta^{ij}\vec{\nabla}\cdot\vec{v}\right)^2 - \zeta(\vec{\nabla}\cdot\vec{v})^2$$

As I have stated above, this is not a fundamentally correct way to represent energy flow. I will now present a better treatment that corrects this difficulty. The new feature that we will need to add to do this is the proper accounting of *heat* and *entropy* in the fluid.

The internal state of a fluid is characterized by thermodynamic functions

$$u, h, p, \rho, T, s$$

I will represent extensive thermodynamic quantities in terms of the value per gram, as I have done with energy and enthalpy above. Another important one of these quantities is entropy, for which I have written

$$s = \frac{S}{\rho V}$$

The heat in the fluid changes according to

$$dQ = T dS$$

In thermodynamics, the temperature T is *defined* by relation (for a uniform system in equilibrium)

$$\frac{1}{T} = \left.\frac{\partial S}{\partial E}\right|_V$$

The derivative is taken at constant volume or, more generally, along a path on which no mechanical work is done on the system. According to this relation, *entropy increases* when heat energy flows from a system at *higher* temperature to a system at *lower* temperature. The second law of thermodynamics states that total entropy must increase, and this defines a direction for heat flow.

When we discuss the thermodynamics of fluids, we treat each local region of a fluid as a macroscopic system that is internally in thermodynamic equilibrium. If adjacent packets of fluid are at different temperatures, heat will flow from the hotter region to the cooler region. To represent this effect, we should add to the energy current a term that reprsents the flow of heat. Using the philosophy of constituitive equations that this equation should contain the minimal number of derivatives, an appropriate form is

$$\vec{J}_\epsilon \Big|_{\text{heat flow}} = -\kappa \vec{\nabla} T$$

With this modification, and setting the right-hand side of the equation equal to zero, the equation of local energy conservation would take the form

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}v^2 + u\right)\right] + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\tfrac{1}{2}v^2 + h\right) + \overleftrightarrow{T}_{visc}\,\vec{v} - \kappa\vec{\nabla}T\right] = 0$$

I will now evaluate the left-hand side of this equation carefully using the Navier-Stokes equation

$$\frac{\partial}{\partial t}\,\rho\left(\tfrac{1}{2}v^2 + u\right) = \frac{\partial\rho}{\partial t}\left(\tfrac{1}{2}v^2 + u\right) + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + \rho\frac{\partial u}{\partial t}$$

I will no longer assume incompressibility, so I will use the full equation of continuity

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0$$

to eliminate $\partial\rho/\partial t$. The thermodynamic relation

$$dE = TdS - p\,dV$$

implies

$$du = Tds - p\,d\left(\frac{1}{\rho}\right) = TdS + \frac{p}{\rho^2}d\rho$$

Then

$$\frac{\partial u}{\partial t} = T\frac{\partial s}{\partial t} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} = T\frac{\partial s}{\partial t} - \frac{p}{\rho^2}\vec{\nabla}(\rho\vec{v})$$

and

$$\frac{\partial v^i}{\partial t} = -(\vec{v}\cdot\vec{\nabla})v^i - \frac{1}{\rho}\nabla^i p - \frac{1}{\rho}\nabla_k T^{ik}_{visc.}$$

Using all of these relations, we can reduce

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}v^2+u\right)\right] = -\vec{\nabla}\cdot(\rho\vec{v})\left[\tfrac{1}{2}v^2+u+\frac{p}{\rho}\right]$$

$$+(\rho v^i)\left(-\vec{v}\cdot\vec{\nabla}\,v^i - \frac{1}{\rho}\nabla^i p - \frac{1}{\rho}\nabla_k T^{ik}_{visc.}\right) + \rho T\frac{\partial s}{\partial t}$$

Next, use the relation

$$h = u + \frac{p}{\varrho}$$

$$dh = T\,dS + \frac{1}{\varrho}\,dp$$

to write

$$\vec{v}\cdot\vec{\nabla}p = \varrho\,\vec{v}\cdot\vec{\nabla}h - \varrho T\,\vec{v}\cdot\vec{\nabla}s$$

Then reduces the above to

$$\frac{\partial}{\partial t}\left[\varrho\left(\tfrac{1}{2}v^2+u\right)\right] = -\vec{\nabla}\cdot(\varrho\vec{v})\left[\tfrac{1}{2}v^2 + \overbrace{u + \frac{p}{\varrho}}^{h}\right]$$

$$- \varrho\,\vec{v}\cdot\vec{\nabla}\left(\tfrac{1}{2}v^2 + h\right) - \nabla^k\left(T^{ik}_{visc}v^i\right) + \nabla^k v^i\,T^{ik}_{visc}.$$

$$+ \varrho T\left(\frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s\right)$$

Finally, add and subtract the term $\vec{\nabla}\cdot(-\kappa\vec{\nabla}T)$. Then

$$\frac{\partial}{\partial t}\left[\varrho\left(\tfrac{1}{2}v^2+u\right)\right] + \vec{\nabla}\cdot\left[\varrho\vec{v}\left(\tfrac{1}{2}v^2+h\right) + \overleftrightarrow{T}_{visc}\cdot\vec{v} - \kappa\vec{\nabla}T\right]$$

$$- \varrho T\left(\frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s\right) + \vec{\nabla}(\kappa\vec{\nabla}T) - \tfrac{1}{2}\left(\nabla^k v^i + \nabla^i v^k\right)T^{ik}_{visc}$$

$$= 0$$

The first line of this equation is our desired equation of energy conservation. If we set this line equal to zero, we obtain the condition for energy to be conserved in the fluid motion. This condition is

$$\varrho T\left(\frac{\partial s}{\partial t}+\vec{v}\cdot\vec{\nabla} s\right) = \vec{\nabla}\left(\kappa \vec{\nabla} T\right) - \frac{1}{2}\left(\nabla^k v^i + \nabla^i v^k\right) T^{ik}_{visc}$$

The constant κ is a phenomenological representation of the rate of heat flow. I will now take this to be, to a first approximation, independent of the temperature, density, and pressure of the fluid. This simplifies the equation slightly. To make the equation simpler and more explicit, I will replace the object T^{ij}_{visc} with the expression for the viscous stresses in the case of an incompressible fluid

$$T^{ik}_{visc} = -\eta\left(\frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i}\right)$$

Then the equation takes the form

$$\varrho T\left(\frac{\partial s}{\partial t}+\vec{v}\cdot\vec{\nabla} s\right) = \kappa \nabla^2 T + \frac{1}{2}\eta\left(\frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i}\right)^2$$

This is a conservation law for the *entropy* of the fluid. The first term on the right-hand side gives the change in entropy due to the *diffusion of heat.* The second term is the term that previously had on the right-hand side of the energy conservation equation. We have now put that term in its correct place, as representing explicitly the *local production of entropy or heat* as a result of the work done by viscous stresses.

The local entropy density of a fluid is a function of the local temperature and pressure. In thermodyamics, we express this through partial derivatives

$$ds = \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp$$

A general thermodynamic (Maxwell) relation states

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

For a fluid like water that is not very compressible, $(\partial V/\partial T)|_p$ is small. Then the second term in ds is small compared to the first and it is a good approximation to drop it. The first term can be written more explicitly as

$$ds = \left(\frac{\partial s}{\partial T}\right)_p dT = \frac{c_p}{T} dT$$

where

$$c_p = T\left(\frac{\partial s}{\partial T}\right)_p$$

the *specific heat* per gram at fixed pressure. With these simplifications, the entropy equation can be written

$$\frac{\partial T}{\partial t} + \vec{v}.\vec{\nabla} T = \frac{\kappa}{\rho c_p} \nabla^2 T + \frac{1}{2}\frac{\eta}{\rho c_p}\left(\frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i}\right)^2$$

or, finally,

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla} T = \chi \nabla^2 T + \frac{1}{2}\frac{\nu}{c_p}\left(\frac{\partial v^l}{\partial x^k} + \frac{\partial v^k}{\partial x^l}\right)^2$$

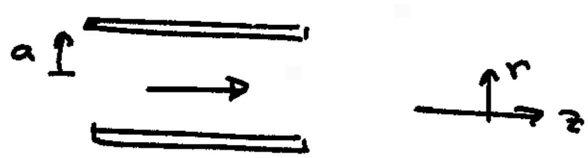
where

$$\chi = \frac{k}{\rho c_p}$$

The parameter χ is the *diffusion constant for temperature or heat.* It has the units $\mathrm{cm}^2/\mathrm{sec}$.

I will now give a simple exercise in the use of this formula. I will ignore the heat generation term for the moment and study the phenomenon of heat transport in a flow. An interesting example is that case of flow in a pipe that is externally heated and becomes hotter along its length,

$$T = T_0 + Az$$



The flow of the fluid should carry colder fluid downstream; this fluid will be heated by diffusion.

I will assume that the fluid maintains a constant density ρ. Then the heating of the fluid does not feed back on the Navier-Stokes equation for the fluid flow. The velocity distribution is therefore Poiseuille flow

$$v_z = V_0\left(1 - \frac{r^2}{a^2}\right)$$

We can now find the temperature distribution in the flow field. The temperature obeys the boundary condition

$$T(r=a) = T_0 + Az$$

As an ansatz, I will take the temperature gradient to be independently of z throughout the fluid. Then we can look for a solution

$$T(r) = T_0 + Az + B(r)$$

with $B = 0$ at $r = a$. Putting this into the diffusion equation, $B(r)$ satisfies

$$V_0 (1 - r^2/a^2) A = \chi \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} B$$

It is straightforward to intergrate this equation,

$$\frac{d}{dr} r \frac{d}{dr} B = \frac{V_0 A}{\chi} (r - r^3/a^2)$$

$$r \frac{d}{dr} B = \frac{V_0 A}{\chi} \left(\frac{1}{2} r^2 - \frac{1}{4} \frac{r^4}{a^2} + C \right)$$

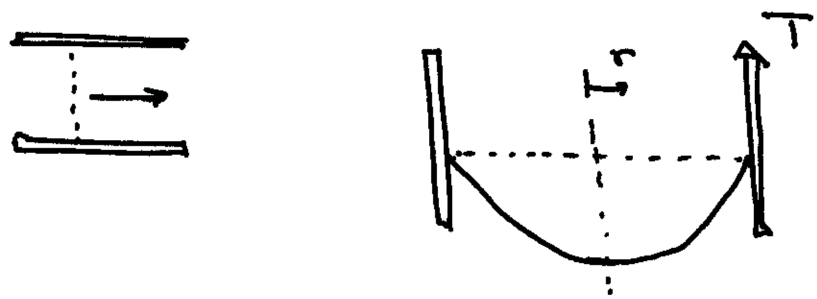
If $C \neq 0$, we obtain a logarithmic singularity in $B(r)$ at $r = 0$. So, if $B(r)$ is to be smooth at $r = 0$, we must have $C = 0$. With this choice, we can integrate further to find

$$B = \frac{V_0 A}{\chi} \left(\frac{1}{4} r^2 - \frac{1}{16} \frac{r^4}{a^2} + D \right)$$

Now D can be fixed so that $B(r)$ vanishes at $r = a$. This gives, finally,

$$T = T_0 + A\left[z - \frac{V_0 a^2}{4\chi}\left(\frac{3}{4} - \frac{r^2}{a^2} + \frac{r^4}{4a^4}\right)\right]$$

Across a cross section of the pipe, the temperature has the form



Note the *linear slope* at the walls

$$\left.\frac{\partial T}{\partial r}\right|_{r=a} = \frac{A V_0 a^2}{4\chi}\left.\left(\frac{2r}{a^2} - \frac{4r^3}{4a^4}\right)\right|_{r=a}$$

or

$$\left.\frac{\partial T}{\partial r}\right|_{r=a} = \frac{A V_0 a}{4\chi}$$

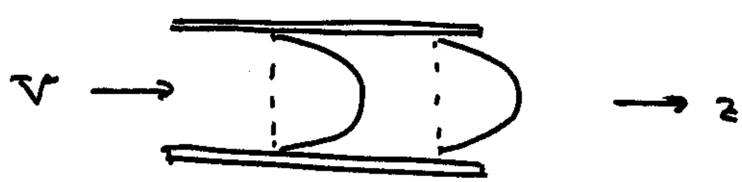
The rate of heat transport into the fluid is proportional to this temperature gradient. The heat extracted from the pipe wall by the fluid per unit area is

$$(-\hat{r})\cdot\vec{J}_Q = (-\hat{r})(-k\vec{\nabla}T) = k\left.\frac{\partial T}{\partial r}\right|_{r=a}$$

Inserting the slope of $T(r)$ at the wall, we find for the heat flux (in units of g cm^2/sec^2 per cm^2 per second)

$$k\,\frac{A\cdot V_0\, a}{4\chi} = \rho c_p \frac{V_0\, a\, A}{4}$$

Another way to represent the temperature distribution is to draw the *isotherms* in the pipe,

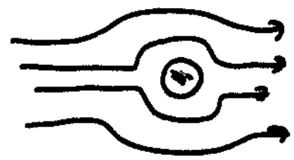


The water stays cooler in the center of the pipe where the fluid velocity is larger and transports the upstream temperature more effectively.

The heating term in the temperature diffusion equation can add to the heat balance. For a flow parallel to $\hat{z}$ and uniform in z, the heating is

$$\frac{\nu}{2c_p}(\vec{\nabla}v_z)^2$$

The effect also makes it tricky to measure the temperature of a moving fluid. The fluid obeys the boundary condition $\vec{v}=0$ at the surface of the thermometer, so there is *shear* and *local heating* just where we are trying to make the measurement.



The new dimensionful parameter χ leads to new dimensionless numbers that characterize flows with heat diffusion. We can consider such flows to be characterized by both the two numbers

$$\text{Reynolds no.} \quad R = \frac{VL}{\nu} \qquad \text{Péclet number} \quad Pe = \frac{VL}{\chi}$$

More commonly, we replace the second of these numbers by the *Prandtl number*

$$P = \frac{\nu}{\chi}$$

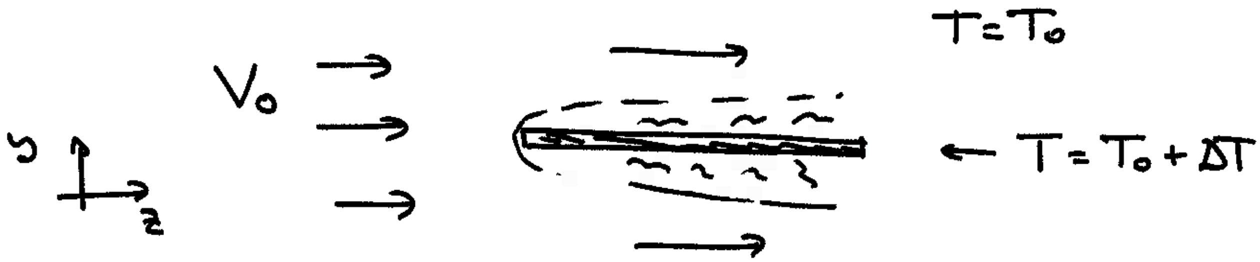
The Prandtl number is an intrinsic property of the fluid. Some representative values are

$$P = \begin{cases} 6.75 & \text{water} \\ 16.6 & \text{alcohol} \\ 0.044 & \text{mercury} \\ 7250. & \text{glycerine} \end{cases}$$

Flows with very different V, L, in which heating by shear can be ignored, have the same form if the values of R and P are the same.

We can apply this theory of heat transport in fluids to the problem of heat conduction in boundary layers. For the laminar boundary layer, a rather complete theory can be given. I assume an incompressible fluid and ignore local heating by friction. I will assume that $R \gg 1$ and $P \sim 1$.

As a model problem, consider the boundary layer on a hot flat plate



We have already worked out the theory of the flow field in this geometry. We find that the fuid velocity is given by a stream function $\psi(y, z)$, with

$$v_z = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial z}$$

The Blasius solution for ψ is

$$\psi = V_0 \sqrt{\frac{2z\nu}{V_0}}\, f(w) \qquad w = y \sqrt{\frac{V_0}{2\nu z}}$$

so that w is of order 1 at a small value of y, or order $1/\sqrt{R}$, that marks the edge of the boundary layer. The asymptotic behaviors of the Blasius function $f(w)$ are

$$f(w) \sim \begin{cases} \frac{1}{2}\alpha w^2 & w \to 0 \\ w - \beta & w \to \infty \end{cases}$$

We can not look for a similar scaling form for $T(y, z)$. Write

$$T = T_0 + \Delta T \, \tau(w)$$

Putting this into the diffusion equation for temperature

$$v_z \frac{\partial T}{\partial z} + v_y \frac{\partial T}{\partial y} = \chi \nabla^2 T$$

with the approximation

$$\nabla^2 T \approx \frac{d^2}{dy^2} T$$

in the boundary layer, we find

$$\frac{\partial \psi}{\partial y} \frac{\partial \tau}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \tau}{\partial y} = \chi \frac{\partial^2}{\partial y^2} \tau$$

More explicitly,

$$V_0\sqrt{\frac{2\nu z}{V_0}}\sqrt{\frac{V_0}{2\nu z}}\,f' \cdot y\sqrt{\frac{V_0}{2\nu z}}\left(-\frac{1}{2z}\right)\tau'$$

$$-\,V_0\sqrt{\frac{2\nu z}{V_0}}\left[\frac{1}{2z}f + y\sqrt{\frac{V_0}{2\nu z}}\left(-\frac{1}{2z}\right)f'\right]\sqrt{\frac{V_0}{2\nu z}}\,\tau'$$

$$= \chi\left(\sqrt{\frac{V_0}{2\nu z}}\right)^2\tau''$$

or

$$-\frac{V_0}{2z}\left[f' w\tau' + (f - wf')\tau'\right] = \chi\frac{V_0}{2\nu z}\tau''$$

or, finally,

$$f\tau' + \frac{1}{P}\tau'' = 0$$

where P is the Prandtl number. This is a second order ordinary differential equation, to be solved with the boundary conditions

$$\tau(w) = 1 \quad \text{at} \quad w = 0$$

$$\tau(w) \to 0 \quad \text{as} \quad w \to \infty$$

We can rewrite this equation as

$$\frac{d}{dw}(\log\tau') = -P f(w)$$

so that

$$\frac{d}{dw}\tau = -A\, e^{-P\int dw\, f(w)}$$

For small values of P, we have linear decrease of τ at the plate. But, for large w,

$$\int dw\, f(w) \sim \frac{1}{2}(w-\beta)^2$$

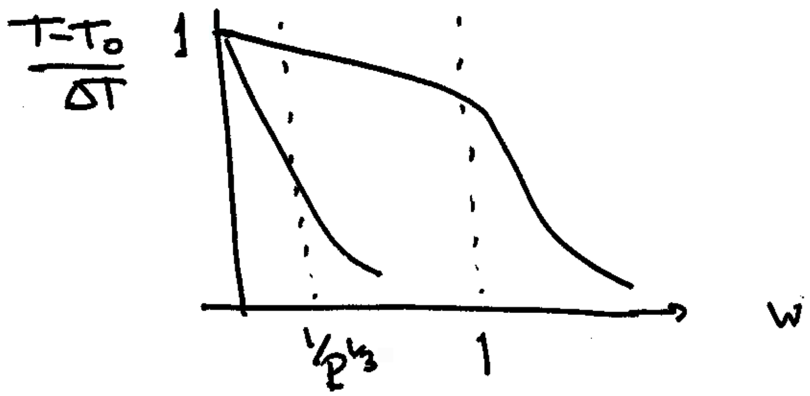
Then the temperature difference τ falls to zero like a Gaussian, characteristic of diffusion in to the region outside the boundary layer. For large values of P, the behavior is different. The temperature different τ falls off already for small w, where

$$f(w) = \frac{1}{2}\alpha w^2 \qquad \int dw\, f(w) = \frac{1}{6}\alpha w^3$$

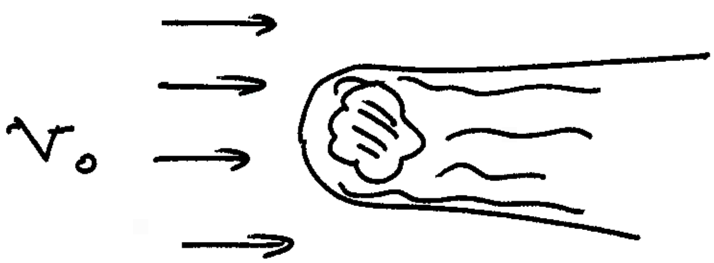
Then, the thickness of the hot region is

$$\Delta y \sim \sqrt{\frac{2\nu z}{V_0}} \cdot \frac{1}{P^{1/3}}$$

Here is a plot of the temperature distribution in the two limits.



This calculation illustrates a more general picture of heat flow in a boundary layer. The flow field has the form that we have discussed in previous lectures



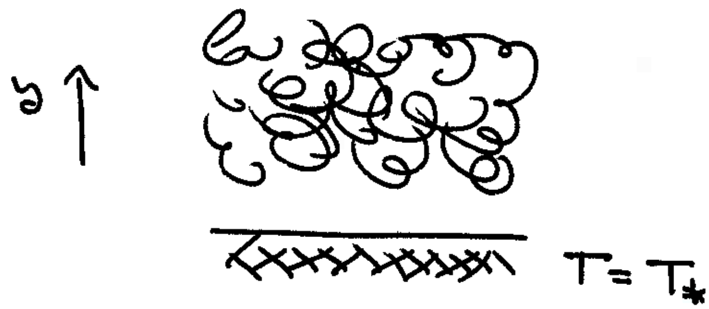
with vorticity only inside a sharp *line of separation.* If the body in the flow is hot, heat diffuses into the boundary layer and is convected downstream. Only the boundary layer and the weak regions are heated signficantly.

If the boundary layer becomes turbulent, we can write formulae for the heat transport in the same way that we previously analyzed the momentum flow in a turbulent situation. We apply dimensional analysis, considering the kinematic viscosity ν to be irrelevant. If P is of order 1, we should also ignore the thermal diffusion constant χ. The dominant mechanism of heat transport will be convection by small eddies. The effective thermal diffusion constant can be estimated as

$$\chi_t \sim v_\ell \cdot \ell$$

This estimate, from dimensional analysis, is the same one that we had for the effective kinematic viscosity ν_t.

Consider first the large-scale temperature distribution in a turbulent flow near a hot wall



We use dimensional analysis to represent dT/dy in the fluid. Let

$$q = \hat{y}\cdot\vec{j}_{\epsilon}\Big|_{y=0}$$

be the heat flow from the wall per cm^2 per second. A useful combination of fluid properties is

$$\frac{q}{\rho c_p} \sim \frac{\dfrac{g\,\mathrm{cm}^2/\mathrm{sec}^2}{\mathrm{cm}^2/\mathrm{sec}}}{g/\mathrm{cm}^3\cdot \mathrm{cm}^2/\mathrm{sec}^2\ \mathrm{degrees}} \sim \frac{\mathrm{cm}}{\mathrm{sec}}\cdot \mathrm{degrees}$$

Using this, we can write a dimensionally correction formula for the temperature gradient in the fluid is

$$\frac{dT}{dy} = \frac{q}{\rho c_p}\,\frac{1}{V_*}\,\frac{1}{y}$$

where $v_*^2 = f/\rho$ as in our previous discussion of turbulence near a boundary wall. If we integrate this equation, we find a logarithmic profile for the temperature

$$T = T_*\left(1 - a\,\frac{q}{\rho c_p V_*}(\log y + c)\right)$$

As before, the $\log y$ dependence is cut off at a distance from the wall sufficiently small that the local Reynolds number becomes of order 1. The final formula is

$$T = T_* \left(1 - \beta \frac{q}{\rho c_p} \frac{1}{b V_*} \log \frac{y V_*}{\nu}\right)$$

where $b = 1/2.4$ is the Landau and Lifshitz constant for the turbulent flow pattern and β is a new constant, depending on the fluid, that must be determined experimentally.

There is one more physical effect that we need to add to our discussion of heat transport in fluids. When heated, fluids typically *expand.* The density decreases, and the gravitational force per unit volume decreases. For fluids like water, this is a small effect,

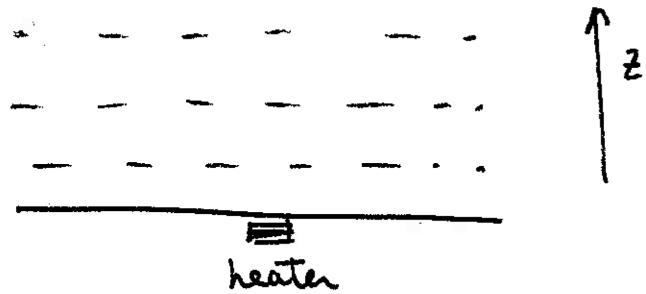
$$\rho = \rho_0 (1 - \alpha \Delta T)$$

with

$$\alpha \sim 10^{-3}/{}^\circ K$$

But, the force of gravity is sufficiently strong that the effect of the density change is nevertheless important. The difference in gravitational forces on elements of fluid of different temperatures can drive a flow of the fluid, and this flow can feed back into thermal transport. This process of *thermal convection* is a very important one. I will now describe of elements of its theory.

To begin our study, I will discuss qualitatively the effects that occur when a fluid is heated locally from below.



Initially, before the heating, the density is ρ_0 and the pressure in the fluid is

$$p = p_0 - \rho_0 g z$$

Each element of fluid experiences an upward force due to the pressure and a downward force due to gravity. In the initial configuration, these are in balance

$$-\vec{\nabla} p = +\rho_0 g \hat{z}$$

$$\rho \vec{g} = -\rho_0 g \hat{z}$$

Now turn on the heater. From from the heater, the original pressure gradient is maintained. As a first approximation, let me posit that the pressure is only a function of z and its unchanged everywhere. But then, in the region around the heater, the temperature of the fluid rises by ΔT. Now the forces are out of balance,

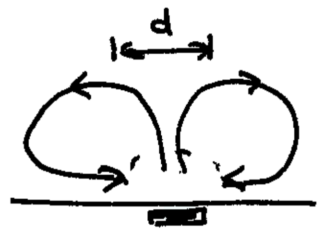
$$-\vec{\nabla} p = +\rho_0 g \hat{z}$$

$$\rho \vec{g} = -\rho_0 (1 - \alpha \Delta T) g \hat{z}$$

There is a net upward force on a fluid element, of magnitude

$$\rho_0 \alpha \Delta T g$$

per unit volume. This force drives a flow



A steady state is reached when the bouyancy force above is balanced by a viscous force due to the shear in this flow. The magnitude of the shear force can be estimated as

$$\eta \nabla^2 v \sim \eta \frac{v}{d^2}$$

where v, d are characteristic velocities and sizes in the flow. The flow carries hot fluid out of the region near the heater and replaces it with colder fluid. The velocity v is limited by the time it takes heat to diffuse out of the wall and warm up these new fluid elements. The time for this diffusion is

$$t \sim \frac{d^2}{\chi} \underset{\text{must be}}{\sim} \frac{d}{v}$$

so we can estimate

$$v \sim \frac{\chi}{d}$$

The flow is strongly driven by the heater if

$$\rho_0 \alpha \Delta T g > \eta \frac{1}{d^2} \frac{\chi}{d}$$

This criterion can be rewritten as

$$Ra = \frac{\alpha g \Delta T d^3}{\nu \chi} > 1$$

where I have defined a new dimensionless number, the *Rayleigh number.*

Heat transport by a flow such as this is called *convection.* If we write the heat transport equation as

$$\frac{\partial}{\partial t} T = -\vec{v} \cdot \vec{\nabla} T + \chi \nabla^2 T$$

the first term on the right is the effect of convection, the second is the effect of diffusion. The time to heat a large region of size ℓ by convection is proportional to ℓ, while the time to heat a region by diffusion is proportional to ℓ^2. So when convection can occur, it is a much more effective means of heat transfer.

Convection is a relatively complicated phenomenon. There do not seem to be any textbook examples of simple convective flows. Convective flows typically require a fluid instability. The computation of the onset of this instability, which occurs at a characteristic value of the Rayleigh number, is a somewhat easier problem. I will now present an example.

First, though, I will introduce an approximation that is commonly used in convection problems. In fluids like water, α is very small. Then, the change in the *inertia* of a fluid with temperature is a very small effect, and the approximation that $\vec{\nabla} \cdot \vec{v} = 0$ is still a good one despite the small density changes. The bouyancy force is the only important mechanism by which the temperature of the fluid affects the fluid flow. This idea is commonly formalized as the *Boussinesq approximation.*

In this approximation, we ignore the thermal expansion of the fluid in the equation of continuity, writing $\vec{\nabla} \cdot \vec{v} = 0$. We also ignore the temperature dependence of the heat diffusion constant χ. However, we keep the bouyancy effect in the Navier-Stokes equation. Write

$$\bar{P} = P - \rho_0 g z$$

and put

$$\rho = \rho_0 (1 - \alpha \Delta T)$$

in the gravitational force term only. Then the Navier-Stokes equation becomes

$$\rho_0 \frac{\partial \vec{v}}{\partial t} + \rho_0 (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \vec{\nabla} p - \rho g \hat{z} + \eta \nabla^2 \vec{v}$$

$$= - \vec{\nabla} \bar{p} + \alpha \rho_0 g \Delta T \, \hat{z} + \eta \nabla^2 \vec{v}$$

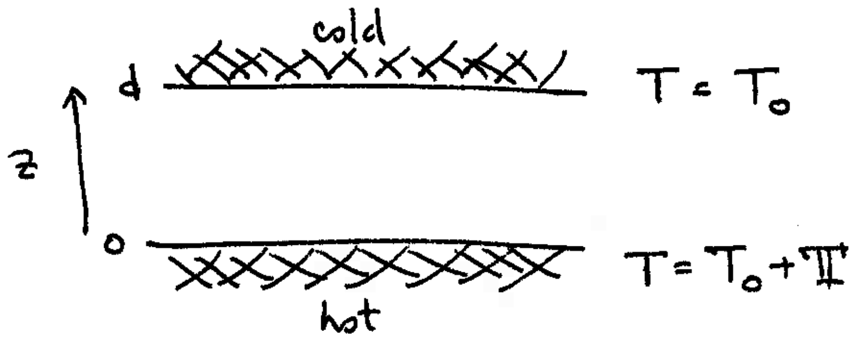
The equations of fluid flow in the *Boussinesq approximation* are then

$$\vec{\nabla} \cdot \vec{v} = 0$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \Delta T = \chi \nabla^2 (\Delta T)$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = - \frac{1}{\rho_0} \vec{\nabla} \bar{p} + \alpha g \Delta T \hat{z} + \nu \nabla^2 \vec{v}$$

We can now use these equations to analyze the instability to convection in a layer of fluid *heated from the bottom* and *cooled from the top.* This is called the *Rayleigh-Bénard problem.* The geometry is



For small temperature difference $\mathbf{T}$, there is a steady-state situation in which there is no fluid flow and heat moves by diffusion. The Boussinesq equations are solved by

$$\vec{v} = 0 \qquad T = T_0 + \mathbf{T}\left(1 - \frac{z}{d}\right)$$

$$\bar{p} = \alpha \rho g \mathbf{T} \, \frac{z\,(2d - z)}{2d}$$

Now I would like to linearize about this solution and examine its stability. I will use the same style of notation as in our earlier discussion of fluid instabilities. Set

$$\vec{v} = \delta\vec{v} \qquad T = (\text{above}) + \delta T \qquad \bar{p} = (\text{above}) + \delta\bar{p}$$

The linearized equations are

$$\vec{\nabla}\cdot\delta\vec{v} = 0$$

$$\frac{\partial}{\partial t}\delta\vec{v} = -\frac{1}{\rho}\vec{\nabla}\delta\bar{p} + \alpha g\,\delta T\,\hat{z} + \nu\nabla^2\delta\vec{v}$$

$$\frac{\partial}{\partial t}\delta T + \delta v_z\left(-\frac{\mathbf{T}}{d}\right) = \chi\nabla^2\delta T$$

Notice that, in the heat transport equation, we pick up a term from $(\vec{v} \cdot \vec{\nabla})T$ that is proportional to **T**. This term is essential in driving the instability.

We can remove the $\vec{\nabla} p$ term from the Navier-Stokes equation by taking the curl. Notice that, if there are thermal gradients, this does not remove the bouyancy effect. We obtain the equation for the vorticity

$$\frac{\partial}{\partial t} \delta\vec{\omega} = \alpha g \vec{\nabla} \delta T \times \hat{z} + \nu \nabla^2 \delta\vec{\omega}$$

Why not take the curl again for good measure?

$$\frac{\partial}{\partial t} \vec{\nabla} \times (\vec{\nabla} \times \delta\vec{v}) = \alpha g \vec{\nabla} \times (\vec{\nabla} \delta T \times \hat{z}) + \nu \nabla^2 \vec{\nabla} \times (\nabla \times \delta\vec{v})$$

In this equation

$$\vec{\nabla} \times (\vec{\nabla} \times \delta\vec{v}) = \vec{\nabla}\, \vec{\nabla} \cdot \delta\vec{v} - \nabla^2 \delta\vec{v} = -\nabla^2 \delta\vec{v}$$

where in the second equality I have used $\vec{\nabla} \cdot \delta\vec{v} = 0$. Another useful rearrangement is

$$\vec{\nabla} \times (\vec{\nabla} \delta T \times \hat{z}) = (\hat{z} \cdot \vec{\nabla}) \vec{\nabla} \delta T - \nabla^2 \delta T\, \hat{z}$$

The z component of this equation

$$\frac{\partial}{\partial t}(-\nabla^2 \delta v_z) = \alpha g \left(\frac{\partial^2}{\partial z^2}\delta T - \nabla^2 \delta T\right) + \nu \nabla^2 (-\nabla^2 \delta v_z)$$

reduces to

$$-\nabla^2 \left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \delta v_z = -\alpha g \nabla_\perp^2 \delta T$$

where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The heat transport equation is

$$\left(\frac{\partial}{\partial t} - \chi \nabla^2\right) \delta T = \frac{\Pi}{d} \delta v_z$$

so we can eliminate δv_z and obtain a single equation for δT

$$-\nabla^2 \left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\left(\frac{\partial}{\partial t} - \chi \nabla^2\right) \delta T = \frac{\alpha g \Pi}{d} (-\nabla_\perp^2) \delta T$$

This is a sixth-order differential equation, but we can solve it if we can make sense of the boundary conditions.

Next, expand the perturbations in Fourier components,

$$\delta T = T(z)\, e^{-i\omega t}\, e^{i\vec{k}\cdot\vec{x}_\perp}$$

$$\delta\vec{v} = \vec{v}(z)\, e^{-i\omega t}\, e^{i\vec{k}\cdot\vec{x}_\perp}$$

where $x_\perp = (x, y)$. The coefficient functions T and v_z are nontrivial functions of the single variable z. The above equation for δT becomes

$$\left(k^2 - \frac{d^2}{dz^2}\right)\left(\frac{-i\omega}{\nu} + k^2 - \frac{d^2}{dz^2}\right)\left(-\frac{i\omega}{\chi} + k^2 - \frac{d^2}{dz^2}\right)T(z) = \frac{\alpha g T'}{\nu\chi d}\, k^2\, T(z)$$

Finally, it is useful to replace k and z by dimensionless variables by extracting powers of the distances d between the upper and lower boundaries,

$$\hat{k} = kd \qquad \hat{z} = \frac{z}{d} \qquad \hat{\omega} = \omega d^2$$

Then we find the following sixth-order ordinary differential equation:

$$\left(\hat{k}^2 - \frac{d^2}{d\hat{z}^2}\right)\left(\frac{-i\hat{\omega}}{\nu} + k^2 - \frac{d^2}{d\hat{z}^2}\right)\left(-i\frac{\hat{\omega}}{\chi} + \hat{k}^2 - \frac{d^2}{d\hat{z}^2}\right)T = \left(\frac{\alpha g T' d^3}{\nu\chi}\right)\hat{k}^2\, T(z)$$

The coefficient on the right-hand side is the dimensionless *Rayleigh number* Ra defined above. Now we are in a position to solve this equation for appropriate eigenvectors satisfying the physical boundary conditions, find the corresponding values of ω, and identify the conditions under which $\operatorname{Im}\omega > 0$.

At this point, I will take a short cut. It can be shown that solutions with $\operatorname{Im}\omega > 0$ arise from solutions with ω^2 real and positive that move to $\omega^2 = 0$ and then to $\omega^2 < 0$

as the thermal gradient is increased. The alternative possibility, which we saw in the stability of Poiseuille flow, in which nonzero real eigenvalues ω acquire a positive imaginary part can be excluded here. A proof is given in the textbook of Drazin and Reid. It follows from this that we can find the stability boundary by looking for eigenfunctions with $\omega = 0$. This greatly simplifies the problem, since the above equation reduces to

$$\left(\hat{k}^2 - \frac{d^2}{d\hat{z}^2}\right)^3 T(\hat{z}) = Ra\ \hat{k}^2 T(\hat{z})$$

The eigenfunctions of this equation are

$$\sin(\kappa \hat{z}) \qquad \cos(\kappa \hat{z})$$

for some appropriate values of κ.

To identify the allowed wavenumbers, we need to define the 6 needed boundary conditions. The physical boundary conditions for this problem are vanishing of the temperature deviation and the three components of velocity at the upper and lower surfaces.

$$T = 0 \quad V_x = V_y = V_z = 0 \quad \text{at} \quad z = 0, d$$

This would seem to be 8 boundary conditions. However, you should notice that the component of (v_x, v_y) that is orthogonal to $\vec{k}$ is not induced by any equation and can be set to zero from the beginning. The component of (v_x, v_y) parallel to $\vec{k}$ is connected to v_z by the equation $\vec{\nabla} \cdot \delta\vec{v} = 0$. On the Fourier components, this reads

$$i\vec{k}\cdot(v_x, v_y) + \frac{\partial}{\partial z} v_z = 0$$

Also, $v_z(z)$ is connected to $T(z)$ by the relation

$$v_z = -\frac{d}{\mathbb{T}}\,\chi\,\nabla^2 T$$

So all of the boundary conditions are satisfied if we choose $T(z)$ so that

$$T = 0 \qquad \frac{d^2}{dz^2}T = 0 \qquad \frac{d^3}{dz^3}T = 0$$

at $z = 0$ and at $z = d$.

Unfortunately, this is not an easy set of boundary conditions to solve. If a second derivative and a third derivative of $T(z)$ must vanish at the boundary, the solution must be a mixture of sines and cosines. The analysis of this problem gets ugly fast.

In his original 1916 paper on this problem, Rayleigh actually defaulted to a simpler problem that is easier to analyze. Instead of considering a fluid with fixed boundaries at $z = 0$ and $z = d$, Rayleigh considered the problem in which the boundaries are membranes that can deform horizontally with the flow. Instead of the boundary conditions above on (v_x, v_y), then, we have the boundary condition that there is no shear force acting on the boundary

$$\frac{\partial}{\partial z} v_{x,y} = 0$$

In terms of $T(z)$, the new boundary conditions are

$$T = 0 \qquad \frac{d^2}{dz^2}T = 0 \qquad \frac{d^4}{dz^4}T = 0 \qquad \text{at } z = 0, d$$

This set of conditions is somewhat difficult to arrange in an experiment, especially on the bottom surface, but it leads to a qualitatively correct picture of the stability boundary in the more realistic situation.

With the new boundary conditions, pure functions of the form

$$\sin(\mu \hat{k} \hat{z}) = \sin(\mu k z)$$

can be eigenfunctions, provided that

$$\mu k d = m\pi \qquad m = 1, 2, \ldots$$

Plugging this form back into the differential equation, we find the condition

$$\left[\hat{k}^2 (1+\mu^2)\right]^3 = Ra\, \hat{k}^2 \quad , \quad \hat{k} = kd$$

The solutions of this equation are

$$\mu = \pm \left[\left(\frac{Ra}{\hat{k}^4}\right)^{\frac{1}{3}} e^{2\pi i n/3} - 1 \right]^{\frac{1}{2}} \qquad n = -1, 0, 1$$

The parameter μ should be real, and $\mu \to -\mu$ gives the same eigenfunction, so we can restrict our attention to the cases

$$\mu = \left[\left[\frac{Ra}{(kd)^4} \right]^{1/3} - 1 \right]^{1/2}$$

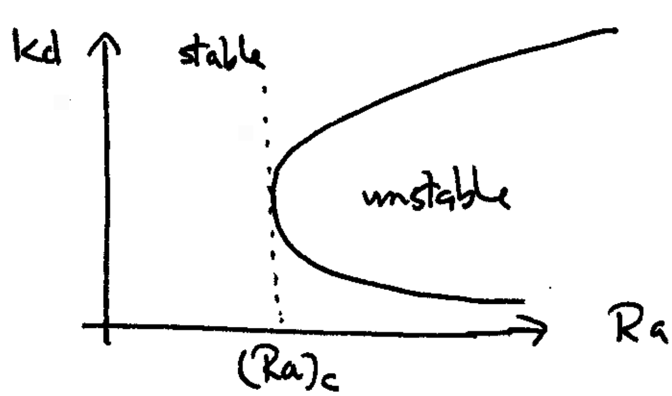
The first instability encountered is that for $m = 1$. This gives as the critical value of the Rayleigh number,

$$\left[\frac{Ra}{(kd)^4} \right]^{1/3} - 1 = \frac{\pi^2}{(kd)^2}$$

or

$$Ra = \frac{\left[(kd)^2 + \pi^2 \right]^3}{(kd)^2}$$

Here is a plot of the boundary in the plane of Ra vs kd:



The boundary curve goes to ∞ as $k \to 0$ and as $k \to \infty$. The region inside the contour to the right is unstable. The function defining the boundary is minimized at

$$(kd)^2 = \frac{\pi^2}{2}$$

so the Rayleigh-Bénard setup is unstable for any Rayleigh number greater than

$$(Ra)_c = \frac{27}{4}\pi^4 = 657.5$$

The problem for experimentally realizable situation with two rigid boundaries was solved in 1928, by Jeffries (of JWKB). He found a curve of a similar shape, with the critical value of the Rayleigh number occurring at

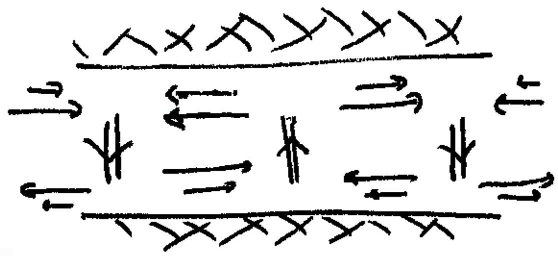
$$(Ra)_c = 1708.$$

The fact that the instability occurs at a nonzero value of k means that the instability leads to a periodic pattern of flows. The eigenfunctions that we found for $T(z)$ corresponds to the perturbations

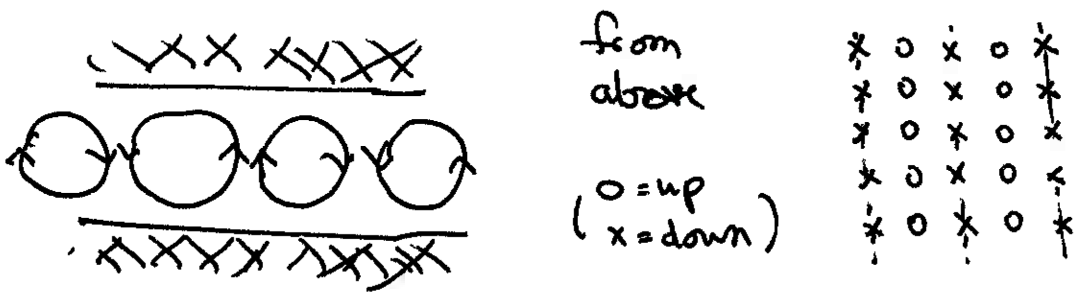
$$\delta T = a \sin\left(\pi \frac{z}{d}\right) e^{ikx}$$

$$\delta v_z = -\frac{d}{\pi}\chi \nabla^2 \delta T = b \sin\frac{\pi z}{d} e^{ikz}$$

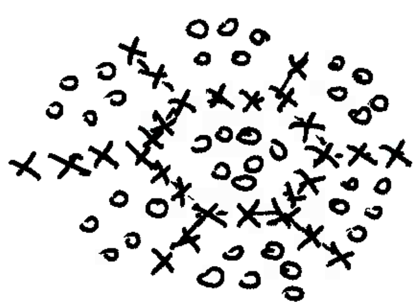
Sketching the δv_z and filling in δv_x so that $\vec{\nabla} \cdot \delta\vec{v} = 0$, we find a flow pattern



The instability then leads to a pattern of rolls

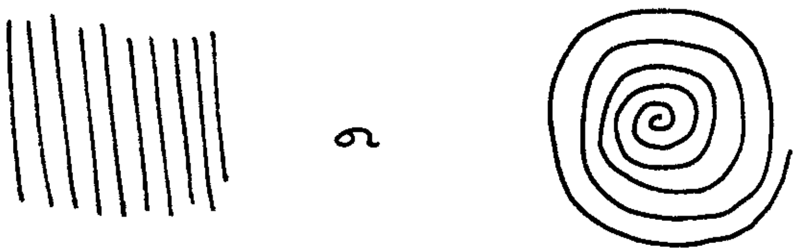


The analysis above fixes the value k for the instability, but it does not fix the direction of $\vec{k}$. In an infinitely extended layer, modes with all directions of $\vec{k}$ go unstable at the same point. In a chamber of finite size, particular directions $\vec{k}$ will be chosen, depending on the shape of the chamber in $(\hat{x}, \hat{y})$ Bénard's experiments (in 1900), as described in Rayleigh's paper, saw 'nearly convex polygons of, in general, 4 to 7 sides'. What is most commonly seen just above the threshold for the instability is a pattern of hexagonal cells

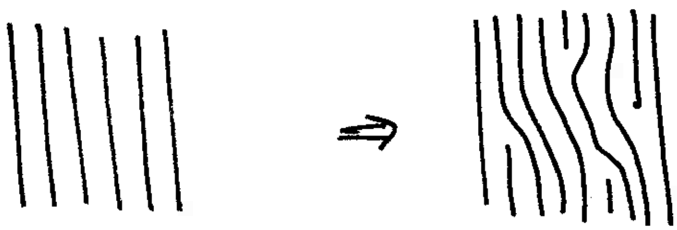


corresponding to exciting instabilities at six values of $\vec{k}$ spaced 60° apart from one another.

It is also possible to set up roll patterns of various structures, for example



Earlier in the course, I referred to experiments of Libchaber and collaborators on the period-doubling route to chaos. These started with the instability to a simple roll pattern that then acquired more structure at higher Rayleigh number, $\mathrm{Ra}/\mathrm{Ra}_c \sim 3$. Ahlers and coworkers have exhibited a wide variety of structures in the dynamics of the Rayleigh-Bénard system at higher Rayleigh number. These include hexagonal to roll transitions and transitions of the roll pattern to what they call *spiral defect chaos*



in which the regular pattern fills up with dislocations. A recent review of patterns in Rayleigh-Bénard convection is found in Bodenschatz, Pesch, and Ahlers, Ann. Rev. Fluid Mech. 32, 709 (2000).