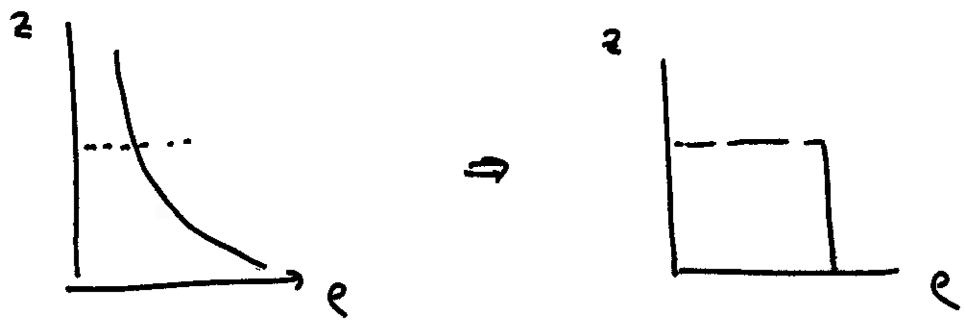
Mar. 2

Flow on a Rotating Sphere

Now that we have derived the main principles of the theory of rotating fluids, I will apply that theory to the motion of atmospheres and oceans. I will discuss only the most simplified models. We will develop some important ideas relevant to geophysics, but our main emphasis will be on fluid dynamics rather than the description of phenomena.

In particular, I will make the idealization that both atmospheres and oceans are described as thin layers of *homogeneous, incompressible fluid* with a fixed, spherical upper boundary. For the oceans, this means that I will treat the top of the ocean as a fixed sphere at $r = R$, the radius of the earth, and that I will ignore temperature variation and effects of convection. For the atmospheres, I will make a slightly more drastic approximation, treating the atmosphere as cut off at its scale height (about 10 km above sea level) and considering the fluid has having constant density and temperture up to that height



I will, further, approximate this layer of fluid as being very *thin*. This is a more reasonable approximation, since the heights of atmospheres and oceans are only a few km and thus tiny compared to the radius of the earth. The layer is bounded above by the fixed surface, which I will take to be at $z = 0$, and bounded below by the ocean bottom, or, for the atmosphere, the terrain.

The Rossby number for such flows is

$$\epsilon = \frac{V}{fL}$$

where V, L are characteristic velocities and distances for the flow and f is the Coriolis parameter

$$f = 2\Omega \cos\Theta = 2\vec{\Omega}\cdot\hat{z}$$

On the earth,

$$\Omega = \frac{2\pi}{1\,\text{day}} = 7.3\times10^{-5}/\text{sec}$$

and so at 45° north latitude $f = 10^{-4}$/sec. Then

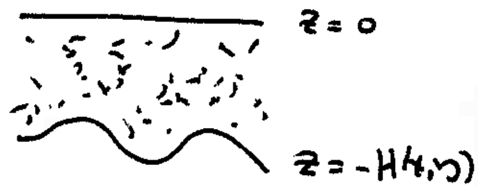
$$\epsilon = \frac{V\ (\text{in units of } 10\,\text{m/sec})}{f\,(\text{in units of } 10^{-4}/\text{sec})\ \ L\ (\text{in units of } 100\,\text{km})}$$

A typical flow on scales greater than 100 km has low Rossby number. This is the reason that low Rossby number flows are often called *geostrophic.* We saw in the previous lecture that, in this regime, $v_z = 0$, or, at least, v_z is as smooth as possible.

If the depth $H(x.y)$ varies and the fluid is incompressible, v_z cannot be absolutely zero. The fluid must stretch and contract in the $\hat{z}$ direction, and so it must contract and stretch in (x, y). Let

$$\vec{\nabla}_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \qquad \vec{v}_\perp = (v_x, v_y)$$

The equation for mass conservation is



$$\vec{\nabla}_\perp \cdot \vec{V}_\perp = -\frac{\partial V_z}{\partial z}$$

Integrate this equation over z, assuming that v_x, v_y vary only by a little over the thin layer.

$$H \cdot \vec{\nabla}_\perp \cdot \vec{V}_\perp = -\int_{-H}^{0} dz\, \frac{\partial V_z}{\partial z} = -\left[V_z(0) - V_z(-H)\right]$$

The top of the fluid is fixed, so

$$V_z(0) = 0$$

The flow at the bottom must allow the fluid to follow the terrain. This requires

$$V_z(-H) = (\vec{\nabla}_\perp \cdot \vec{\nabla})(-H) = -(\vec{V}_\perp \cdot \vec{\nabla}_\perp)\, H = -\frac{D}{Dt} H$$

The last step follows because $H(x, y)$ has no explicit dependence on t. Then, finally,

$$\vec{\nabla}_\perp \cdot \vec{V}_\perp = -\frac{1}{H} \frac{DH}{Dt}$$

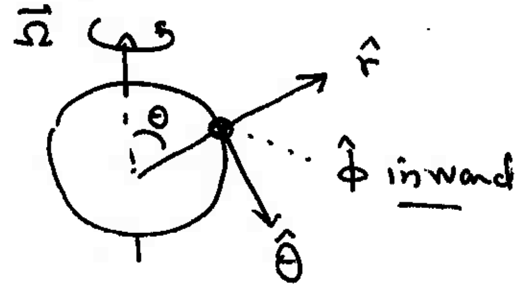
In the following discussion, I will consider v_x, v_y to be functions of (x, y) only. I will ignore v_z. However, we will not be able to treat the fluid as an incompressible 2-dimensional fluid. Instead, we will use the above equation for $(\vec{\nabla}_\perp \cdot \vec{v}_\perp)$.

With these approximations, write the Navier-Stokes equation, ignoring viscosity, for fluid flow on a sphere in a rotating coordinate system. To begin,

$$\frac{D\vec{v}}{Dt} = -2\vec{\Omega}\times\vec{v} - \vec{\nabla}\varphi$$

It will be useful to rewrite this equation is spherical polar coordinats. For this we need to define the unit vectors of spherical coordinates

$$\hat{r} \quad \hat{\theta} \quad \hat{\phi}$$



The nonzero derivatives of these unit vectors are

$$\hat{\theta}\cdot\vec{\nabla}\,\hat{r} = \frac{\hat{\theta}}{r} \qquad \hat{\theta}\cdot\vec{\nabla}\,\hat{\theta} = -\frac{\hat{r}}{r} \qquad \hat{\theta}\cdot\vec{\nabla}\hat{\phi} = 0$$

$$\hat{\phi}\cdot\vec{\nabla}\,\hat{r} = \frac{\cancel{\sin\theta}}{r\cancel{\sin\theta}}\,\hat{\phi} \qquad \hat{\phi}\cdot\vec{\nabla}\hat{\theta} = \frac{\cos\theta}{r\sin\theta}\,\hat{\phi} \qquad \hat{\phi}\cdot\vec{\nabla}\hat{\phi} = -\frac{(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})}{r\sin\theta}$$

Using these relations to act derivatives on

$$\vec{v} = v_r\,\hat{r} + v_\theta\,\hat{\theta} + v_\phi\,\hat{\phi}$$

we see that

$$\left(\frac{Dv}{Dt}\right)_r = \frac{\partial v_r}{\partial t} + \vec{v}\cdot\vec{\nabla}\, v_r - \frac{v_\theta^2}{r} - \frac{v_\phi^2}{r}$$

$$\left(\frac{Dv}{Dt}\right)_\theta = \frac{\partial v_\theta}{\partial t} + \vec{v}\cdot\vec{\nabla}\, v_\theta + \frac{v_\theta v_r}{r} - \frac{v_\phi^2}{r}\cot\theta$$

$$\left(\frac{Dv}{Dt}\right)_\phi = \frac{\partial v_\phi}{\partial t} + \vec{v}\cdot\vec{\nabla}\, v_\phi + \frac{v_\phi v_r}{r} + \frac{v_\phi v_\theta}{r}\cot\theta$$

Now approximate these equations by ignoring v_r. The radial component of the Navier-Stokes equation is now written as

$$-\frac{v_\theta^2}{r} - \frac{v_\phi^2}{r} = 2\Omega v_\phi \sin\theta - \frac{\partial}{\partial r}\varphi$$

We can integrate this equation to determine φ. The content of that statement is that a vertical pressure difference will develop that cancels the radial component of the centrifugal and Coriolis forces. Thus, we will be able to neglect these forces when we solve for the purely horizontal motions.

The $\hat{\theta}$ and $\hat{\phi}$ components of the Navier-Stokes equation are

$$\frac{D}{Dt} v_\theta - \frac{v_\phi^2}{R}\cot\theta = f v_\phi - \frac{1}{R}\frac{\partial}{\partial\theta}\varphi$$

$$\frac{D}{Dt} v_\phi + \frac{v_\phi v_\theta}{R}\cot\theta = -f v_\theta - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\varphi$$

wehre I have put $r = R$, the radius of the earth. and

$$f = 2\Omega\cos\theta$$

I have also used the 2-dimensional convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{v_\theta}{R}\frac{\partial}{\partial \theta} + \frac{v_\phi}{R\sin\theta}\frac{\partial}{\partial \phi}$$

In a 2-dimensional flow, the vorticity points normal to the 2-dimensional plane and can be treated as a scalar. In this problem $\vec{\omega} = \omega\hat{r}$, where

$$\omega = \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\phi\right) - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} v_\theta$$

We can obtain an equation for ω by taking the curl of the Navier-Stokes equations. In particular, we apply to the Dv_θ/Dt and Dv_ϕ/Dt equations the derivatives in the definition of *omega*. The pressure terms in the equations must cancel out because the curl of a gradient is zero. Explicitly,

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(-\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\varphi\right)\right] - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(-\frac{1}{R}\frac{\partial}{\partial\theta}\varphi\right)$$

$$= -\frac{1}{R\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\varphi + \frac{1}{R\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\varphi = 0$$

The f terms give

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(-f v_\theta\right) - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(f v_\phi\right)$$

$$= f\left(-\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}v_\phi\right) - \frac{v_\theta}{R}\frac{df}{d\theta}\right.$$

$$= -f\left(\vec{\nabla}_\perp\cdot\vec{v}_\perp\right) - \frac{v_\theta}{R}\frac{df}{d\theta}$$

The $\partial/\partial t$ tems give, simply,

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\phi}{\partial t}\right) - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial v_\theta}{\partial t}\right) = \frac{\partial}{\partial t}\omega$$

The $\vec{v}\cdot\vec{\nabla}$ terms are somewhat complicated. It is simplest to reduce these by going through a step with commutators of differential operators

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left[\frac{V_\theta}{R}\frac{\partial}{\partial\theta}+\frac{V_\phi}{R\sin\theta}\frac{\partial}{\partial\phi}\right]V_\phi - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{V_\theta}{R}\frac{\partial}{\partial\theta}+\frac{V_\phi}{R\sin\theta}\frac{\partial}{\partial\phi}\right]V_\theta$$

$$= \left(\frac{V_\theta}{R}\frac{\partial}{\partial\theta}+\frac{V_\phi}{R\sin\theta}\frac{\partial}{\partial\phi}\right)\omega + \left[\frac{1}{R}\left(\frac{\partial}{\partial\theta}+\cot\theta\right), \left(\frac{V_\theta}{R}\frac{\partial}{\partial\theta}+\frac{V_\phi}{R\sin\theta}\frac{\partial}{\partial\phi}\right)\right]V_\phi$$

$$- \left[\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}, \left(\frac{V_\theta}{R}\frac{\partial}{\partial\theta}+\frac{V_\phi}{R\sin\theta}\frac{\partial}{\partial\phi}\right)\right]V_\theta$$

giving finally

$$= \vec{V}_\perp\cdot\vec{\nabla}_\perp\omega + \frac{1}{R^2}\frac{\partial V_\theta}{\partial\theta}\frac{\partial V_\phi}{\partial\theta} + \frac{1}{R^2\sin\theta}\frac{\partial V_\phi}{\partial\theta}\frac{\partial V_\phi}{\partial\phi} - \frac{\cos\theta}{R^2\sin^2\theta}V_\phi\frac{\partial V_\phi}{\partial\phi} + \frac{1}{R^2\sin^2\theta}V_\theta V_\phi$$

$$- \frac{1}{R^2\sin\theta}\frac{\partial V_\theta}{\partial\phi}\frac{\partial V_\theta}{\partial\theta} - \frac{1}{R^2\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\frac{\partial V_\theta}{\partial\phi} - \frac{\cos\theta}{R^2\sin^2\theta}V_\theta\frac{\partial V_\theta}{\partial\phi}$$

We can form this into $(\vec{v}_\perp\cdot\vec{\nabla}_\perp)\omega + (\vec{\nabla}_\perp\cdot\vec{v}_\perp)\omega$

$$= (\vec{V}_\perp\cdot\vec{\nabla}_\perp)\omega + \left[\frac{1}{R}\left(\frac{\partial V_\theta}{\partial\theta}+\cot\theta\, V_\theta+\frac{\partial V_\phi}{\sin\theta\,\partial\phi}\right)\frac{1}{R}\left(\frac{\partial V_\phi}{\partial\theta}+\cot\theta\, V_\phi-\frac{\partial V_\theta}{\sin\theta\,\partial\phi}\right)\right]$$

$$- \frac{\cot\theta}{R^2}V_\theta\frac{\partial V_\phi}{\partial\theta} + \frac{\sin^2\theta}{R^2\sin^2\theta}V_\theta V_\phi - \frac{\cot\theta}{R^2}V_\phi\frac{\partial V_\theta}{\partial\theta} - 2\frac{\cos\theta}{\sin^2\theta}V_\phi\frac{\partial V_\phi}{\partial\phi}$$

The terms with $\cot\theta$ give

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{V_\phi V_\theta}{R}\cot\theta\right) - \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(-\frac{V_\phi^2}{R}\cot\theta\right)$$

$$= \frac{1}{R^2}\cot\theta\frac{\partial}{\partial\theta}(V_\theta V_\phi) - \frac{1}{R^2}V_\theta V_\phi + \frac{\text{[redacted]}}{R^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}(V_\phi^2)$$

and cancel out the second line of the previous equation. After this cancellation, the ω equation becomes

$$\frac{\partial \omega}{\partial t} + \vec{v}_\perp \cdot \vec{\nabla}_\perp \omega + (\vec{\nabla}_\perp \cdot \vec{v}_\perp)\,\omega = -f\,(\nabla_\perp \cdot v_\perp) - \frac{v_\theta}{R}\frac{df}{d\theta}$$

The last term here is

$$\frac{v_\theta}{R}\frac{df}{d\theta} = (\vec{v}_\perp \cdot \vec{\nabla}_\perp)\, f$$

Then

$$\left(\frac{\partial}{\partial t} + \vec{v}_\perp \cdot \vec{\nabla}_\perp\right)(\omega + f) = -(\vec{\nabla}_\perp \cdot \vec{v}_\perp)\,(\omega + f)$$

or, using the equation for $(\vec{\nabla}_\perp \cdot \vec{v}_\perp)$ above,

$$\frac{D}{Dt}(\omega + f) = \left(\frac{1}{H}\frac{DH}{Dt}\right)(\omega + f)$$

Then, finally, we have the simple relation

$$\frac{D}{Dt}\left(\frac{\omega + f}{H}\right) = 0$$

This equation is called the *conservation of potential vorticity.* The scalar ω is the local vorticity in the flow. Along a streamline, the argument of D/Dt is conserved. Thus, along a streamline, ω plays off against f and $H(x, y)$.

For localized fluids at a fixed latitude, we can set f to be constant

$$f = f_0 = 2\Omega \cos\theta_0 \qquad \theta_0 = \frac{\pi}{2} - (\text{latitude})$$

Then

$$\frac{D}{Dt}\left(\frac{\omega}{H}\right) = 0$$

In a larger region, we can approximate f by a linear function. Let

$$x = R\phi \sin\theta_0 \qquad \text{East-West distance}$$

$$y = R(\theta_0 - \theta) \qquad \text{North-South distance}$$

Then

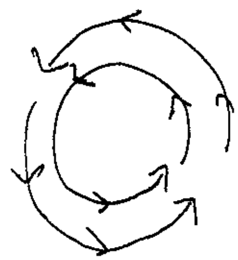
$$f = f_0 + \beta y$$

with

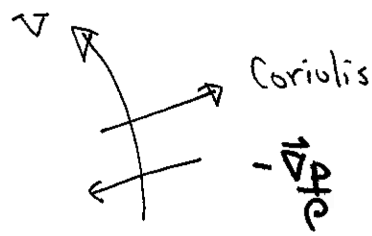
$$\beta = \frac{2\Omega \sin\theta_0}{R} = 1.6 \times 10^{-13}/\text{cm sec} \quad \text{at } 45^\circ\text{N}$$

$$= \frac{f_0}{100 \text{ km}}$$

In the northern hemisphere, $f(\theta)$ increases as we go north toward the pole. So, for H approximately constant, a large mass of air that moves south will spin up, acquiring $\omega > 0$. This produces a *cyclonic flow*



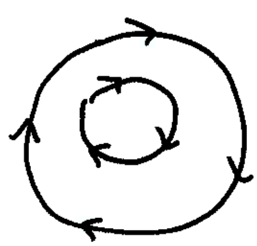
Typically, the energy of this system is increased by nonlinear effects of heat convection, due to vertical motions and density differences that are not included in this analysis. The result is a storm system. The simple cyclone balances Coriolis force against a pressure gradient



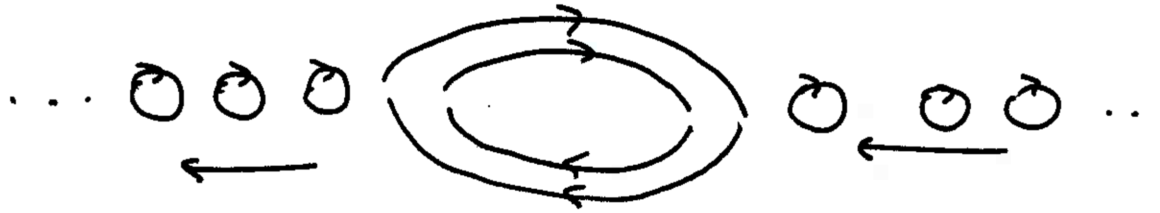
so the center of the cyclone has *low* pressure. If the wind speed increases, there is a centrifugal force that must also be balanced by pressure. Then

$$\left|\frac{\vec\nabla p}{\rho}\right| = \frac{v_t^2}{r} + fv$$

An *anticyclone* is a flow in the opposite direction, with *high pressure* in the center

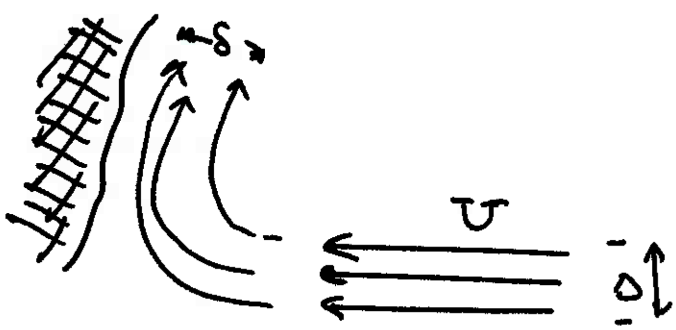


Similar analysis applies to other planets, which might have more complicated atmospheric dynamics. Jupiter has a huge anticylone called the *Great Red Spot.* Small anticyclones nucleate to the left of the spot, travel around the planet, and are absorbed to the right of the spot.



Using Voyager data, one can measure the potential vorticity $(\omega + f)$ over the Great Red Spot. From these measurements, it follows that H varies by a factor of 2 from north to south in this system. Thus, the Great Red spot is confined to a shallow fixed layer at the top of the Jovian atmosphere. For must more discussion of the atmospheric dynamics of Jupiter, see T. E. Dowling, Ann. Rev. Fluid Mech. 27, 293 (1995).

Now we can consider some quantitative applications to geophysical problems. Consider first an ocean with a significant westerly current. When the current meets a continent (for example, at the Asian boundary of the Pacific Ocean), it turns north. In doing so, it must acquire negative vorticity. We can work out the effect of this.



Let the initial fluid velocity be U, over a region of width Δ. The mass flow is

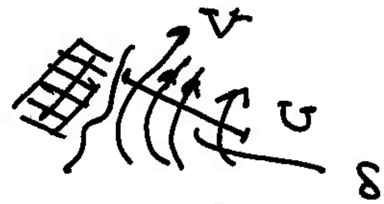
$$\frac{Q}{\rho H} = U \cdot \Delta$$

At the outer edge of the current, the pressure is constant, so if the flow is inviscid we know from Bernoulli's theorem that the velocity is constant: $v = U$. The current initial has vorticity $\omega = 0$, but as the flow is pushed north, it must acquire vorticity

$$\omega = -f$$

So the current must speed up near the coast. As the flow goes north, the coastal velocity V must increase and the width of the flow δ must decrease so that

$$-\omega = \frac{V-U}{\delta} = f(\theta)$$



The mass flow is fixed:

$$q = \frac{Q}{\rho H} = U\cdot\Delta = \frac{U+V}{2}\cdot\delta$$

We can solve these two equations for V and δ,

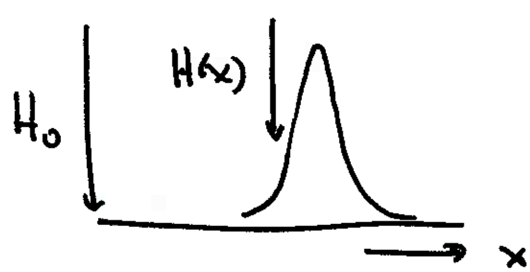
$$V = \left(2U\,[f\Delta + U]\right)^{\frac{1}{2}}$$

If the first term dominates, then

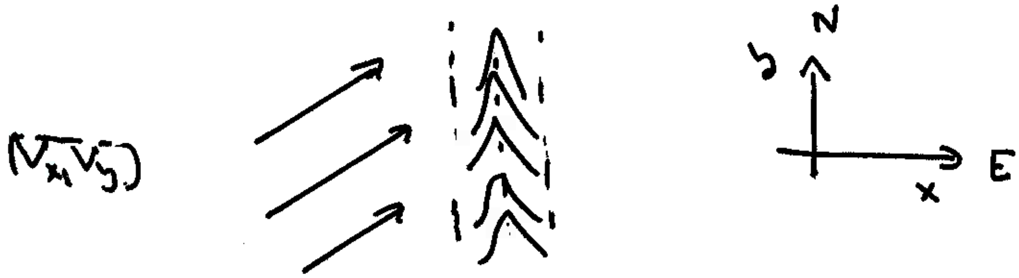
$$V \approx [2fq]^{\frac{1}{2}} \qquad \delta = \left[\frac{2q}{f}\right]^{\frac{1}{2}}$$

with both quantities determined by the mass flow.

Next, we can consider some examples in which H varies. Consider first a local region, in which f can be taken to be constant, in which a wind blows over a mountain range. For simplicity, I will assume that the mountain range is aligned north-south. The depth H varies with the east-west distance x as



Note that H *decreases* as the terrain rises. Let the wind velocity vector to the west of the mountains be $(\bar{V}_x, \bar{V}_y)$. I will now work out how the wind direction changes as the wind crosses the mountains.



The initial vorticity is zero. Along the streamlines,

$$\frac{f+\omega}{H} = \text{constant}$$

so

$$\omega = \left(\frac{H(x)}{H_0} - 1\right) f_0$$

Now (v_x, v_y) depends only on x, so

$$\omega = \frac{dv_y}{dx} = -\left(1 - \frac{H(x)}{H_0}\right) f_0$$

Over the mountains, $H(x) < H_0$, so the right-hand side is negative. We can integrate it to find

$$v_y(x) = \bar{V}_y - \int dx\, f_0 \left(1 - \frac{H(x)}{H_0}\right)$$

To determine v_x, we can use the equation

$$\vec{\nabla}_\perp \vec{V}_\perp = \frac{dV_x}{dx} = -\frac{1}{H}\frac{D}{Dt}H = -\frac{1}{H}V_x\frac{dH}{dx}$$

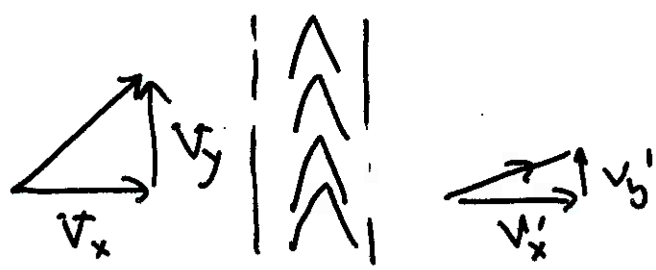
or

$$H\frac{dV_x}{dx} + V_x\frac{dH}{dx} = \frac{d}{dx}(V_x H) = 0$$

which gives

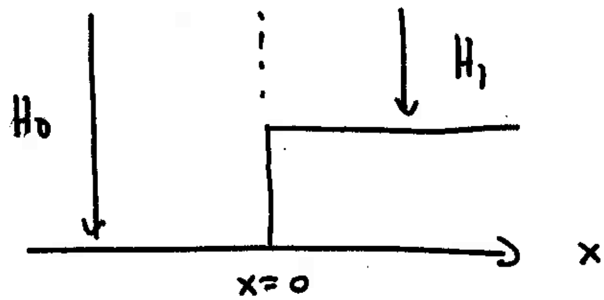
$$V_x(x) = V_x' \cdot \frac{H_0}{H(x)}$$

So, v_x speeds up over the mountains, to preserve the volume of the approximately incompressible fluid, and then returns to its original value on the other side. However, v_y decreases continually. Then the final picture is



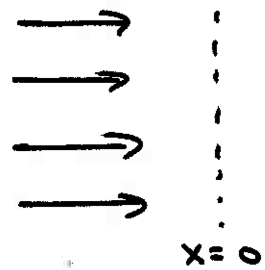
This problem is discussed further in Batchelor, and you will find there some additional interesting comments. In principle, a localized mountain can be a source of vorticity in flow. However, if the flow is sufficiently slow, so that the Rossby number is small, the mountain will have a Taylor-Proudman column above it. Then the flow avoids the mountain and is unaffected.

Next, consider an example in which β cannot be neglected. As a simple setting, I will analyze the effect of a step change in H going from west to east.



I will consider a flow coming directly in from the west, that is

$(U, 0)$



Just to the right of the step, at $x = 0^+$, we have by conservation of mass

$$v_x = U \cdot \frac{H_0}{H_1}$$

and by conservation of potential vorticity

$$\omega = -f\left(1 - \frac{H_1}{H_0}\right)$$

$$= -(f_0 + \beta y)\left(1 - \frac{H_1}{H_0}\right)$$

In the region $x > 0$, H is constant, so

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

and the 2-dimensional fluid is incompressible. Then we can analyze the situation using a stream function ψ that satisfies

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

Then, just to the right of $x = 0$,

$$\psi = U \frac{H_0}{H_1} \cdot y$$

Batchelor now suggests the following trick: For $x > 0$, H is constant, so

$$\frac{D}{Dt}(\omega + f) = 0$$

This means that $(\omega + f)$ is constant along streamlines. The streamlines are the lines of constant ψ, so $(\omega + f)$ is given uniquely as a function of ψ. To find the relation, we need only consider the values of these quantities at $x = 0^+$. Then

$$\omega + f = f \cdot \frac{H_1}{H_0} = (f_0 + \beta y)\frac{H_1}{H_0} = f_0 \frac{H_1}{H_0} + \beta\left(\frac{H_1}{H_0}\right)^2 \frac{\psi}{U}$$

Now that we have found $(\omega + f)$ as a function of ψ, we can use this relation for $x > 0$. However, the vorticity ω is also given in terms of ψ by

$$\omega = -\nabla^2 \psi$$

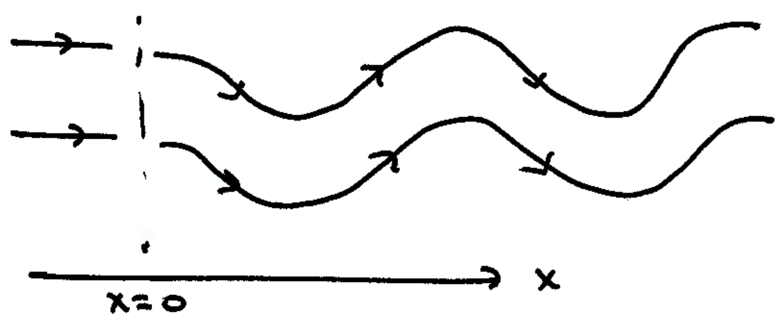
This implies

$$\nabla^2\psi = (f_0+\beta y) - f_0\frac{H_1}{H_0} - \frac{\beta}{U}\left(\frac{H_1}{H_0}\right)^2\psi$$

This equation implies that ψ oscillates as a function of x with the wavenumber

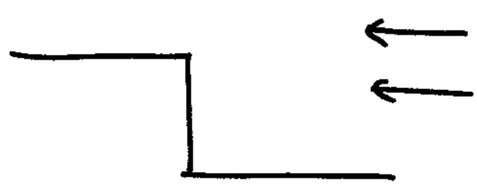
$$k^2 = \frac{\beta}{U}\left(\frac{H_1}{H_0}\right)^2$$

The final picture of the flow is



The flow balances Coriolis forces against pressure. The Coriolis force pushes the horizontal flow to the right (south). but, as the flow goes to the south, the Coriolis force decreases, pressure becomes more dominant, and the flow is pushed back north again.

A similar flow at a western boundary



obeys the same equations as above, but now with $U < 0$. The resulting differential equation has exponential solutions. The Coriolis force pushes the flow north, and it becomes stronger as the flow moves further north.

As a final example, I will consider the small oscillations of an atmosphere in a region with constant H large enough that we must consider $\beta \neq 0$. The flow is again described by a velocity potential. The vorticity is given by

$$\omega = -\nabla^2 \psi$$

Decompose ψ into Fourier modes

$$\psi(x,y,t) = \Psi e^{-i\underset{\sim}{\omega} t + i k_x x + i k_y y}$$

$$\omega = k^2 \Psi$$

vorticity

The equation

$$\frac{D}{Dt}(\omega + f) = 0$$

takes the linearized form

$$\frac{\partial}{\partial t}\omega + v_y \frac{df}{dy} = 0$$

or

$$-i\underset{\sim}{\omega} k^2 \Psi - i k_x \Psi \cdot \beta = 0$$

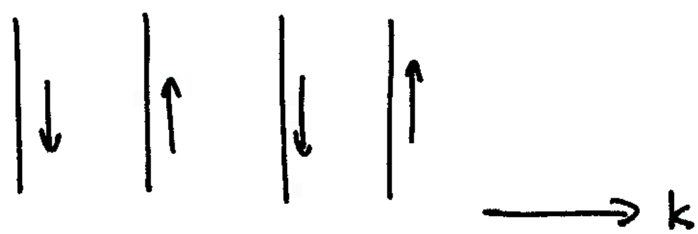
From this, we can read off the dispersion relation

$$\underset{\sim}{\omega} = -\frac{\beta k_x}{k^2}$$

This is a somewhat unusual dispersion relation, and, indeed, these waves, called *Rossby waves*, have many unusual properties. The form of the velocity field is

$$(v_x, v_y) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right) = i\,(k_y, -k_x)\,\psi$$

The wave crests are then normal to (k_x, k_y), so the fluid velocity is parallel to the wave crests. For example, for $\vec{k} = (k, 0)$,



The phase velocity is

$$v_p = \frac{\omega}{k} = -\frac{\beta}{k}$$

so the crests of the waves move to the west. In fact, all modes with the same value of k^2 have crests that move to the west at the some speed

$$\frac{\beta}{(k^2)^{1/2}}$$

On the other hand, the group velocity is positive, so wavepackets viewed as a whole move to the east.

These Rossby waves produce striking patterns in the earth's atmosphere and are even more visible in the atmospheres of Jupiter and other gaseous planets.