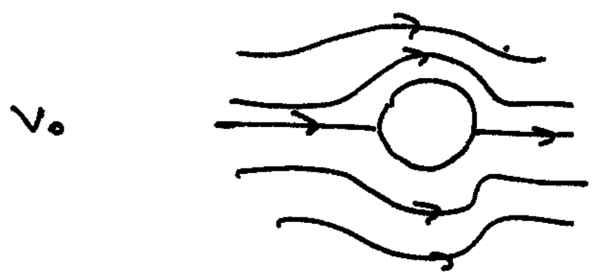
Physics 211 - Final Exam
Solutions

1.) a.) The incompressible irrotational flow around a sphere of radius a at rest is

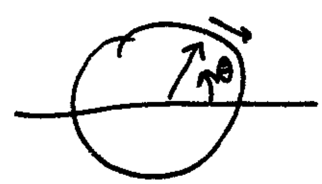


$$\vec{V} = \vec{\nabla}\left[\vec{V}_0 \cdot \vec{r} + \frac{1}{2} \frac{\vec{V}_0 \cdot \hat{r}}{r^2} a^3 \right]$$

$$= \vec{V}_0 \left[1 + \frac{1}{2} \frac{a^3}{r^3} \right] - \frac{3}{2} \vec{V}_0 \cdot \hat{r} \, \hat{r} \frac{a^3}{r^3}$$

The tangential velocity on the sphere is

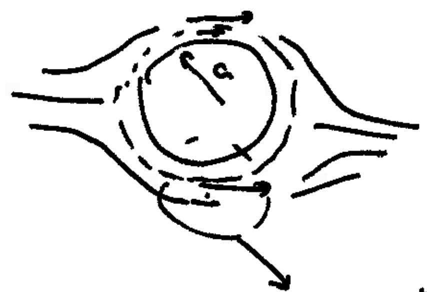
$\vec{V}_{\parallel}(r=a) = \vec{V}(r=a)$ since $V_{\perp} = 0$ at $r=a$



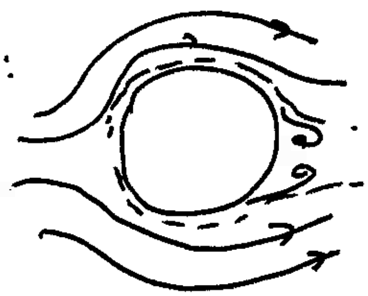
$$= \frac{3}{2} \vec{V}_0 - \frac{3}{2} \vec{V}_0 \cdot \hat{r} = \frac{3}{2} V_0 \sin\theta \, (-\hat{\theta})$$

indeed $\hat{r} \cdot \vec{V}_{\parallel} = 0$ at $r=a$

Just on the sphere, $\vec{V} = 0$ if $\nu \neq 0$. You are supposed to assume that $V_{\parallel} \to 0$ linearly over a thin boundary layer of fixed thickness $\left(\frac{\nu a}{V_0}\right)^{1/2}$

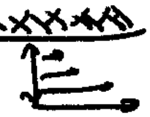


(in reality:



)

then $\left(\frac{\nu a}{V_0}\right)^{1/2}$



$$\frac{\overrightarrow{\text{Force}}}{\text{area}} = \eta \frac{\partial \vec{V}_{\parallel}}{\partial r} = \eta \left(\frac{V_0}{\nu a}\right)^{1/2} \vec{V}_{\parallel}$$

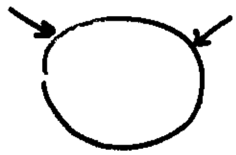
$$= \rho \left(\frac{\nu V_0}{a}\right)^{1/2} \frac{3}{2} V_0 \sin\theta \; (-\hat{\theta})$$

$$= \frac{3}{2} \rho \left(\frac{\nu}{a}\right)^{1/2} V_0^{3/2} \sin\theta \cdot (-\hat{\theta})$$

$$\hat{z} \cdot \frac{\overrightarrow{\text{Force}}}{\text{area}} = \frac{3}{2} \rho \left(\frac{\nu}{a}\right)^{1/2} V_0^{3/2} \sin^2\theta$$

units? $\quad g/cm^3 \left(\frac{cm^2}{sec\, cm}\right)^{1/2} \left(cm/sec\right)^{3/2} = g\, cm/sec^2 / cm^2$ ✓

b.) In irrotational incompressible flow, the pressure is symmetrical on the front and back of the sphere. The pressure drop across the boundary layer is negligible. So



$$\hat{z} \cdot \left(\frac{\overrightarrow{\text{Force}}}{\text{area}} \text{ due to pressure} \right) = 0$$

c)

$$\int_{-1}^{1} d\cos\theta \, \sin^2\theta = \frac{4}{3}$$

$$\text{Drag} = \hat{z} \cdot (\overrightarrow{\text{total Force}})$$

$$= \int_{-1}^{1} d\cos\theta \; 2\pi a^2 \left(\hat{z} \cdot \frac{\overrightarrow{\text{Force}}}{\text{area}} \right)$$

$$= \int_{-1}^{1} d\cos\theta \; 2\pi a^2 \; \frac{3}{2} \rho \left(\frac{\nu}{a} \right)^{1/2} V_0^{3/2} \sin^2\theta$$

$$= 4\pi \rho \, \nu^{1/2} a^{3/2} V_0^{3/2}$$

2.) a.) The temperature distribution obeys

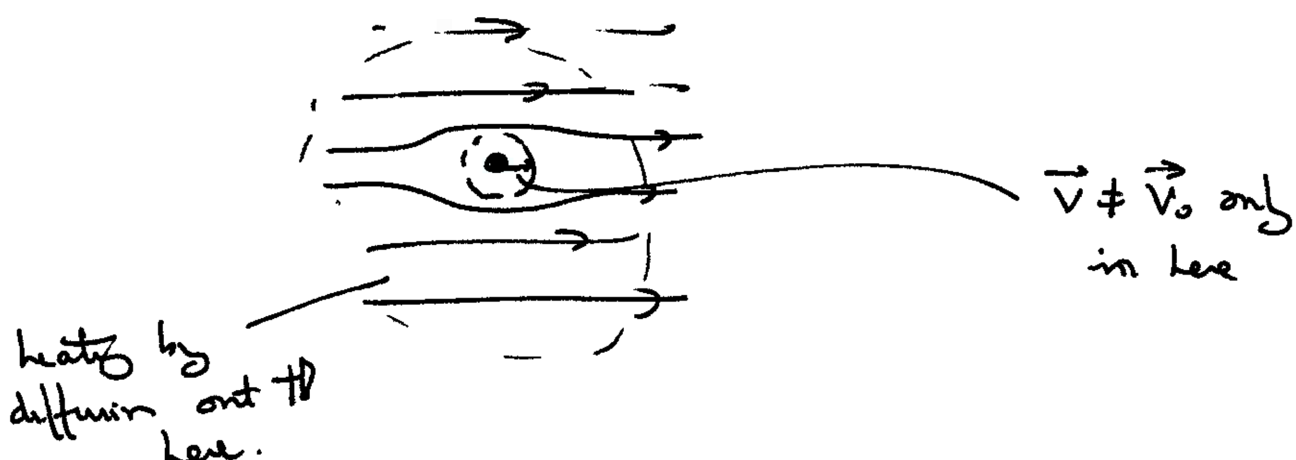
$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}\, T = \chi\, \nabla^2 T + (\text{heating})$$

In this problem, T is t-independent and there is negligible steam

so

$$\vec{v}\cdot\vec{\nabla}\, T = \chi\, \nabla^2 T \qquad \left(\begin{array}{c}\text{away from the}\\ \text{source of heat}\end{array}\right)$$

b.) If $\chi \gg \nu$, the probe is surrounded by a boundary layer of size $\sim \nu^{1/2}$, but heat easily diffuses out to a much larger radius.



So in the region relevant to the temperature distribution

$\vec{v} = \vec{v}_0$.

c.) If $\vec{v}_0 = 0$ $\nabla^2 T = 0$ away from the probe

$$T = \frac{\text{const}}{|\vec{x}|}$$ is a solution to this equation.

d.) For $\vec{v}_0 \neq 0$

$$\vec{v}_0 \cdot \vec{\nabla} T = \chi \nabla^2 T + (\text{const}) \cdot \delta(\vec{x})$$

let $$T(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} T(k)$$

then

$$i\vec{k}\cdot\vec{v}_0 \, T(k) = -k^2 T(k) + C$$

$$T(k) = \frac{C}{k^2 + i\vec{k}\cdot\vec{v}_0/\chi}$$

$$= \frac{C}{\left(k + i\frac{\vec{v}_0}{2\chi}\right)^2 + \frac{v_0^2}{4\chi^2}}$$

transforming back,

$$T = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \frac{c}{(\vec{k}+i\frac{\vec{V}_0}{2\chi})^2 + \frac{V_0^2}{4\chi^2}}$$

$$= \int \frac{d^3k'}{(2\pi)^3}\, e^{i\vec{k}'\cdot\vec{x}}\, e^{i(-i\frac{\vec{V}_0}{2\chi})\cdot\vec{x}} \frac{c}{k'^2 + V_0^2/4\chi^2}$$

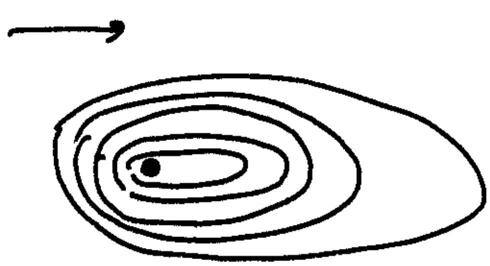
$$= \frac{c}{4\pi|\vec{x}|}\, e^{-\frac{V_0}{2\chi}|x|}\, e^{+\frac{\vec{V}_0\cdot\vec{x}}{2\chi}}$$

Now analogy to the solution of part (c)

$$T(\vec{x}) = \frac{T}{|\vec{x}|}\, e^{-\frac{V_0}{2\chi}(|x| - z)}$$

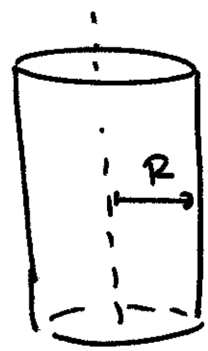
This solution falls off as $\frac{1}{|z|}$ on the line $x=y=0$, $z>0$ and otherwise falls off exponentially. The isotherms look like

$\vec{V}_0$



which is physically correct.

3) a.) The unperturbed situation is a cylinder of fluid at density ρ



The Newtonian potential satisfies

$$\nabla^2 \Phi = 4\pi G \rho$$

Then

$$\underbrace{\int d^2a}_{\text{surface}} \; \hat{n}\cdot\vec{\nabla}\Phi = 4\pi G \underbrace{\int d^3x}_{\substack{\text{enclosed}\\ \text{volume}}} \rho = 4\pi G \, L \, \pi R^2 \rho$$

$$= 2\pi R L \, g$$

$$\text{so} \qquad g = 2\pi G \rho R$$

b.) Inside the cylinder

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{d\Phi}{dr}$$

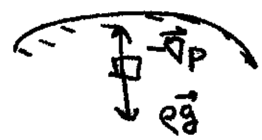
according to the above, inside the cylinder

$$\vec{g} = -\vec{\nabla}\Phi = -2\pi G\rho r \,\hat{r}$$

then

$$\frac{dp}{dr} = -2\pi G\rho^2 r$$

$$p = -\pi G\rho^2 r^2 + (\text{const})$$



such that $p=0$ at $r=R$

$$p = \pi G\rho^2(R^2-r^2)$$

c.) The Navier - Stokes equation is

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\,\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi = -\vec{\nabla}W$$

the linearized version is

$$\frac{\partial}{\partial t}\,\delta\vec{v} = -\vec{\nabla}\,\delta W$$

For z-independent perturbations

$$\frac{\partial}{\partial t}\delta v_r = -\frac{\partial}{\partial r}W$$

$$\frac{\partial}{\partial t}\delta v_\phi = -\frac{1}{r}\frac{\partial}{\partial \phi}W$$

The equation of continuity is

$$\vec{\nabla}\cdot\delta\vec{v} = 0 \qquad \text{since } \rho = \text{const.}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r} r\,\delta v_r + \frac{1}{r}\frac{\partial}{\partial \phi}\delta v_\phi = 0$$

d.) In Fourier space, the above equations become

$$-i\omega\, v_r = -\frac{d}{dr}W$$

$$-i\omega\, v_\phi = -\frac{1}{r}\, imW$$

$$\frac{1}{r}\frac{d}{dr}(r v_r) + \frac{im}{r} v_\phi = 0$$

Eliminating v_ϕ between the last ~~two~~ equations gives

$$-i\omega\,\frac{1}{r}\frac{d}{dr}(r v_r) = (-i\omega)\,\frac{-im}{r} v_\phi = -\frac{im}{r}\left(-\frac{im}{r}W\right)$$

or

$$-i\omega\,\frac{1}{r}\frac{d}{dr}(r v_r) = -\frac{m^2}{r^2}W$$

e.) Now using the first equation above

$$\frac{1}{r}\frac{d}{dr} r \left(-\frac{d}{dr}W\right) = -\frac{m^2}{r^2} W$$

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{d}{dr}W\right) - \frac{m^2}{r^2}W = 0$$

Look for solutions of this equation of the form

$$W \sim r^{\alpha}$$

then $\quad \alpha^2 - m^2 = 0 \quad \Rightarrow \quad \alpha = \pm m$

so
$$W = Ar^{m} + Br^{-m}$$

a solution regular at the origin has $\quad W = Ar^{m}$

f.) Away from the surface of the fluid, the density is either ρ (inside) or 0 outside. In either case, $\delta\rho = 0$. Then $\delta\Phi$ obeys

$$\nabla^2 \delta\Phi = 0$$

In Fourier modes

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\Phi - \frac{m^2}{r^2}\Phi = 0$$

This is the same equation as in part (e). It leads to

$$\Phi(r) = \begin{cases} B r^{m} & \text{inside} \\ C r^{-m} & \text{outside} \end{cases}$$

g.) We still have unknowns $A\ B\ C\ \eta$ for each Fourier mode. We now need to eliminate $A\ B\ C$ in terms of η.

To relate B, C to η, insist (as in problem #2 of problem set 1) that the force of gravity is continuous across the boundary at

$$r = R + \eta(\phi, t)$$

Inside:

$$\vec{g} = -2\pi G \rho r\, \hat{r} - \vec{\nabla}\, \delta\Phi^{(in)}$$

Outside

$$\vec{g} = -2\pi G \rho R^2 \frac{1}{r}\, \hat{r} - \vec{\nabla}\, \delta\Phi^{(out)}$$

the $\vec{\nabla}\delta\Phi$ terms have both radial and azimuthal components, eg.

$$\delta\Phi^{(in)} = \sum_m{}' B_m r^m e^{im\phi}$$

$$-\vec{\nabla}\, \delta\Phi^{(in)} = -\sum_m{}' \left(+im\, \hat{\phi}\, B_m r^{m-1} + m B_m r^{m-1}\, \hat{r}\right) e^{im\phi}$$

Match at $\quad r = R + \sum_m \eta_m e^{im\phi}$

$$\frac{1}{r} = \frac{1}{R} - \frac{1}{R^2} \sum_m \eta_m \, e^{im\phi}$$

to 1st order in small quantities:

$$- 2\pi G\rho R \,\hat{r} \; - \; \sum_m 2\pi G\rho \,\eta_m e^{im\phi}\,\hat{r}$$

$$- \sum_m \left(im\hat{\phi} B_m R^{m-1} e^{im\phi} + m B_m R^{m-1} e^{im\phi} \hat{r} \right)$$

$$= \quad - 2\pi G\rho R \hat{r} \; + \sum_m 2\pi G\rho \,\eta_m \, e^{im\phi} \,\hat{r}$$

$$- \sum_m \left(im\hat{\phi}\, C_m R^{-m-1} e^{im\phi} - m C_m R^{-m-1} e^{im\phi} \hat{r} \right)$$

Match the coefficients of $\hat{\phi}$

$$im B_m R^{m-1} = im C_m R^{-m-1}$$

or $\quad C_m = B_m R^{2m}$

Match the coefficients of $\hat{r}$

$$2\pi G\rho \eta_m + m B_m R^{m-1} = - 2\pi G\rho \eta_m - m C_m R^{-m-1}$$

$$4\pi G\rho \eta_m = - 2m \, B_m R^{m-1}$$

so $$B_m = -2\pi G\rho \frac{1}{m} R^{-m+1} h_m$$

Finally, the pressure is given by

$$P = \pi G\rho^2 (R^2 - r^2) + \delta p \qquad \delta p = \rho \delta W - \delta \Phi$$

$$= \pi G\rho^2 (R^2 - r^2) + \sum_m (\rho A_m - \rho B_m) r^m e^{im\phi}$$

at $\quad r = R + \sum_m h_m e^{im\phi} \quad$ the surface of the fluid

$$0 = -2\pi G\rho^2 R \sum_m h_m e^{im\phi}$$

$$+ \sum_m (\rho A_m - \rho B_m) R^m e^{im\phi}$$

or

$$0 = -2\pi G\rho^2 R h_m + \rho A_m R^m + 2\pi G\rho^2 \frac{1}{m} R h_m$$

$$A_m R^m = 2\pi G\rho \left(\frac{m-1}{m}\right) R h_m$$

h.) Finally, use

$$-i\omega\,(-i\omega\,\eta) = -i\omega\, v_r = -\frac{d}{dr} W$$

$$\omega^2\, \eta = \frac{d}{dr} W$$

in Fourier modes

$$\omega^2\, \eta_m = m\, A_m\, R^{m-1}$$

$$= 2\pi G \rho\, (m-1)\; \eta_m$$

so

$$\omega^2 = 2\pi G \rho\, (m-1)$$

i) For $m = 1$, the perturbation is

surface at $r = R$ $\rightarrow$ surface at $r = R + \eta \cos(\phi - \phi_0)$

(for $\phi_0 = 0$)

this is a simple translation of the cylinder, which should be neutral in energy.

For $m = 0$, this is an overall gravitational collapse of the cylinder. This is the Jeans instability, which is present for small k_z.