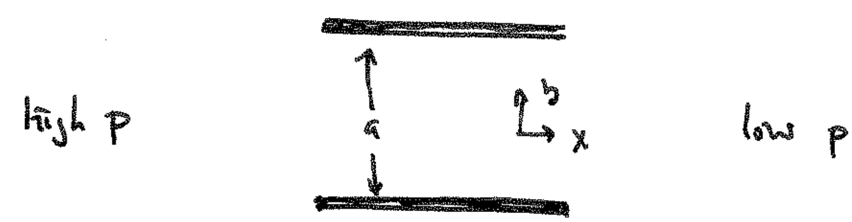
Physics 211 – Final Exam
Solutions

1.) a.) $(-T^{xy})$ is the force exerted on a surface normal to $\hat{y}$ by a fluid with a shear $\frac{dv^x}{dy} \neq 0$. This force is positive if $\frac{dv^x}{dy} > 0$ and <u>reverses</u> if $\frac{dv^x}{dy}$ or $V^x(y)$ changes sign, by parity. So T^{xy} should be an <u>odd</u> function of $\frac{dv^x}{dy}$.

b.) For this situation:



the generalization of the Navier Stokes equation is

$$\frac{\partial \vec{V}^k}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}^k = -\frac{1}{\rho}\vec{\nabla}^k p - \frac{1}{\rho}\nabla^i T^{ik}_{(\text{shear})}$$

With $\quad T^{xy} = -\eta\,\sigma_0 \sinh^{-1}\left(v^x{}_{,y}/\sigma_0 \right)$

For $k=x$, steady flow, v^x a fnct of y only

$$0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\sigma_0 \frac{\partial}{\partial y}\sinh^{-1}\left(\frac{\partial v^x}{\partial y}\Big/\sigma_0 \right)$$

Write $\quad -\frac{\partial p}{\partial x} = G$

$$\frac{\partial}{\partial y}\sinh^{-1}\left(\frac{\partial v^x}{\partial y}\Big/\sigma_0 \right) = -\frac{1}{\eta\sigma_0}G$$

$$\sinh^{-1}\left(\frac{\partial v^x}{\partial y}\Big/\sigma_0 \right) = -\frac{1}{\eta\sigma_0}G\,(y-c)$$

The velocity $v^x(y)$ should have a maximum at $y = a/2$

so $\quad c = a/2$.

$$\frac{\partial v^x}{\partial y} = -\sigma_0 \sinh\left(\frac{G}{\eta\sigma_0}\left(y-\frac{a}{2}\right)\right)$$

$$v^x = -\frac{\eta\sigma_0^2}{G}\cosh\left(\frac{G}{\eta\sigma_0}\left(y-\frac{a}{2}\right)\right) + \text{const.}$$

$v^x = 0$ at $y=0$, $y=a$, so

$$v^x = \frac{\eta\sigma_0^2}{G}\left(\cosh\left[\frac{G}{\eta\sigma_0}\frac{a}{2}\right] - \cosh\left[\frac{G}{\eta\sigma_0}\left(y-\frac{a}{2}\right)\right]\right)$$

The mass flow per unit length in z is

$$Q = \rho \int_0^a dy \; v^x \qquad \bar{y} = y - a/2$$

$$= 2\rho \; \frac{\eta\sigma_0^2}{G} \int_0^{a/2} d\bar{y} \left[\cosh \frac{G}{\eta\sigma_0}\frac{a}{2} - \cosh\left(\frac{G}{\eta\sigma_0}\bar{y}\right)\right]$$

$$= 2\frac{\rho\eta\sigma_0^2}{G}\left\{\frac{a}{2}\cosh\frac{G}{\eta\sigma_0}\frac{a}{2} - \frac{\eta\sigma_0}{G}\sinh\frac{G}{\eta\sigma_0}\frac{a}{2}\right\}$$

$$Q = \frac{\rho\eta\sigma_0^2}{G}\, a\left[\cosh\left(\frac{Ga}{2\eta\sigma_0}\right) - \frac{2\eta\sigma_0}{Ga}\sinh\left(\frac{Ga}{2\eta\sigma_0}\right)\right]$$

c.) For Poiseuille flow, the generalized Navier Stokes equation becomes

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{1}{\rho}\frac{1}{r}\frac{\partial}{\partial r}(r\,T^{zr})$$

$$-\frac{\partial}{\partial z}P = G = -\frac{1}{r}\frac{\partial}{\partial r}\, r\;\eta\sigma_0 \sinh^{-1}\left(\frac{\partial v^z}{\partial r}/\sigma_0\right)$$

$$\frac{Gr^2}{2} = -r\;\eta\sigma_0\sinh^{-1}\left(\frac{\partial v^z}{\partial r}/\sigma_0\right)$$

$$\frac{G}{2\eta\sigma_0}r = -\sinh^{-1}\left(\frac{\partial v^z}{\partial r}/\sigma_0\right)$$

Again, we should have $\frac{\partial v^z}{\partial r} = 0$ at $r=0$

$$\frac{\partial v^z}{\partial r} = -\sigma_0 \sinh\frac{G}{2\eta\sigma_0}r$$

$$v^z = -\frac{2\eta\sigma_0^2}{G} \cosh\left(\frac{G}{2\eta\sigma_0} r\right) + \text{Const.}$$

& since $v^z(r=A) = 0$

$$v^z = \frac{2\eta\sigma_0^2}{G}\left[\cosh\frac{GA}{2\eta\sigma_0} - \cosh\frac{G}{2\eta\sigma_0} r\right]$$

The mass flow is

$$Q = \rho \int_0^A dr\; 2\pi r\; \frac{2\eta\sigma_0^2}{G}\left[\cosh\frac{GA}{2\eta\sigma_0} - \cosh\frac{G}{2\eta\sigma_0} r\right]$$

$$= \frac{2\rho\eta\sigma_0^2}{G}\left[\pi A^2 \cosh\frac{GA}{2\eta\sigma_0} - \frac{4\pi\sigma_0\eta}{G} A \sinh\frac{GA}{2\eta\sigma_0} + \frac{8\pi\eta^2\sigma_0^2}{G^2}\cosh\frac{GA}{2\eta\sigma_0}\right]$$

d.) For the flow between parallel plates, as G becomes large

$$v^x \to \frac{\eta\sigma_0^2}{G}\,\frac{1}{2}\left[e^{\frac{Ga}{2\eta\sigma_0}} - e^{\frac{G}{\eta\sigma_0}(y - a/2)}\right] \quad (\text{for } y > \tfrac{a}{2})$$

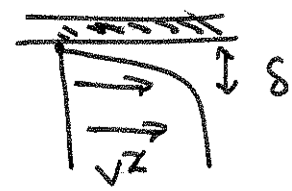
$\delta = a - y$

$y = a - \delta$

$$= \frac{\eta\sigma_0^2}{2G} e^{\frac{Ga}{2\eta\sigma_0}}\left[1 - e^{-\frac{G}{\eta\sigma_0}\delta}\right]$$

so the velocity is zero near the wall but tends to a constant exponentially in a distance

$$\delta = \frac{\eta \sigma_0}{G}$$



All of the shear occurs in this boundary layer.

For the pipe, in the soln above, the boundary layer has thickness

$$\delta = \frac{2\eta\sigma_0}{G}.$$

e.) The temperature distribution obeys.

$$\frac{\partial T}{\partial t} + v^x \frac{\partial}{\partial x} T = \kappa \nabla^2 T$$

For the steady-state situation in this problem

$$T = \text{fcn of } y \text{ only.} + Bx$$

$$v^x B = \kappa \nabla^2 T = \kappa \frac{d^2}{dy^2} T$$

$$\frac{d^2T}{dy^2} = \frac{B}{\kappa} \frac{\eta\sigma_0^2}{G} \left[\cosh \frac{Ga}{2\eta\sigma_0} - \cosh \frac{G}{\eta\sigma_0}(y - a/2)\right]$$

$$\frac{dT}{dy} = \frac{B}{\kappa} \frac{\eta\sigma_0^2}{G} \left[(y - a/2) \cosh\frac{Ga}{2\eta\sigma_0} - \frac{\eta\sigma_0}{G} \sinh \frac{G}{\eta\sigma_0}(y - a/2)\right]$$

$$+ \text{(const)}$$

$$T = \frac{B}{K}\frac{\eta\sigma_0^2}{G}\left\{\frac{1}{2}(y-a/2)^2\cosh\frac{Ga}{2\eta\sigma_0} - \left(\frac{\eta\sigma_0}{G}\right)^2\cosh\frac{G}{\eta\sigma_0}(y-a/2)\right\}$$

$$+ A'y + B'$$

Now impose $T = T_0 + Bx$ on $y = 0, a$

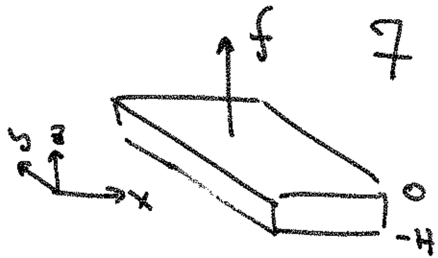
$$T = T_0 + Bx$$

$$+ \frac{B}{K}\frac{\eta\sigma_0^2}{G}\left\{-\frac{y(a-y)}{2}\cosh\frac{Ga}{2\eta\sigma_0} + \left(\frac{\eta\sigma_0}{G}\right)^2\left[\cosh\frac{Ga}{2\eta\sigma_0} - \cosh\frac{G}{\eta\sigma_0}(y-a/2)\right]\right\}$$

Note ~~that~~ $T < (T_0 + Bx)$

in the ~~interior~~. Cooler fluid is convected up from lower x.

2.) a.) The Navier Stokes equation for this situation is



$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})v = -\frac{1}{\rho_0}\vec{\nabla}p - g\hat{z}\,\frac{\rho(y)}{\rho_0} - \vec{f}\times\vec{v}$$

The last term is the Coriolis force. Let $\vec{v} \parallel \hat{x}$ and time-independent; then

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = 0$$

The $\hat{z}$ equation is then

$$-\frac{\partial}{\partial z}p - \rho_0 g\,(1-\alpha T(y)) = 0$$

so

$$p = -g\rho_0 z\,(1-\alpha(T_0 + Ay)) + p_0$$

i.e. p increases linearly with depth, with a coefficient decreasing with y. Then

$$\frac{\partial p}{\partial y} = g\rho_0\alpha A z$$

low pressure
$\updownarrow$
high pressure

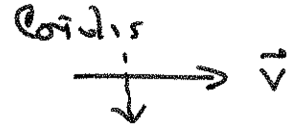
increasing with depth.

The $\hat{y}$ equation is

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} - f v_x = 0$$

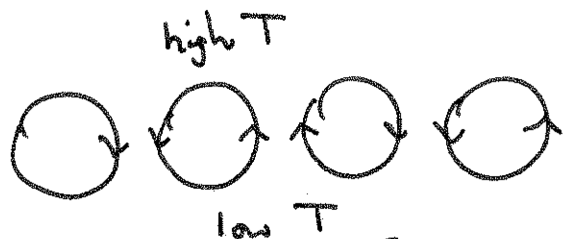
so

$$v_x = -\frac{g\alpha A}{f} z$$



The Coriolis force balances the pressure force.

b) There is a potential instability in this configuration. If a velocity is set up in the y direction, this gives convection that decreases the temperature gradient and so decreases the pressure gradient. But, the Coriolis force is still present. This has the right sign to drive the flow in y. So we get cells



superposed on the overall flow $\Longrightarrow$

This is the baroclinic instability, which is responsible for weather. The rest of this problem gives a highly oversimplified theory of this instability.

c.) The basic equation for this situation are:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho_0}\vec{\nabla}p - \frac{1}{\rho_0} g\hat{z}\,\rho(T) - \vec{f}\times\vec{v}$$

$$\vec{\nabla}\cdot\vec{v} = 0$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T = 0$$

linearizing about $\vec{v} = \hat{x}\,(+V_0(z))$

$$p = p_0 - g\rho_0 z\,(1 - \alpha\,(T_0 + Ay))$$

$$T = T_0 + Ay$$

Write $\vec{v} = +V_0\hat{x} + \delta\vec{v}$ etc.

<u>then</u>:

$$\frac{\partial}{\partial t}\delta\vec{v} + V_0(z)\frac{\partial}{\partial x}\delta\vec{v} + \delta v^z\frac{\partial}{\partial z}V_0\,\hat{x}$$

$$= -\frac{1}{\rho_0}\vec{\nabla}\,\delta p - g\hat{z}(-\alpha\,\delta T) - \hat{f}\times\delta\vec{v}$$

$$\vec{\nabla}\cdot\delta\vec{v} = 0$$

$$\frac{\partial}{\partial t}\delta T + V_0\frac{\partial}{\partial z}\delta T + \delta v^y . A = 0$$

d) Approximate v^x, v^y = indep of z. The equation $\vec{\nabla}\cdot\delta\vec{v} = 0$ becomes

$$\frac{\partial}{\partial x}\delta v^x + \frac{\partial}{\partial y}\delta v^y + \frac{\partial}{\partial z}\delta v^z = 0$$

But, if the top and bottom of the fluid are fixed,

$$\delta v^z = 0 \quad \text{at} \quad z = 0, -H$$

then

$$\int_{-H}^{0} dz \left(\frac{\partial}{\partial x}\delta v^x + \frac{\partial}{\partial y}\delta v^y\right) = H\cdot\left(\frac{\partial}{\partial x}\delta v^x + \frac{\partial}{\partial y}\delta v^y\right)$$

$$= -\int_{-H}^{0} dz \, \frac{\partial}{\partial z}\delta v^z$$

$$= -\left[\delta v^z(0) - \delta v^z(-H)\right] = 0$$

so

$$\Lambda = \frac{\partial}{\partial x}\delta v^x + \frac{\partial}{\partial y}\delta v^y = 0$$

Now look at the x and y Navier Stokes equations

$$\left(\frac{\partial}{\partial t} + \overline{V_0(z)}\frac{\partial}{\partial z}\right)\delta v^x = f\,\delta v^y - \frac{1}{\rho_0}\frac{\partial}{\partial x}\delta p$$

$$\left(\frac{\partial}{\partial t} + \overline{V_0(z)}\frac{\partial}{\partial z}\right)\delta v^y = -f\,\delta v^x - \frac{1}{\rho_0}\frac{\partial}{\partial y}\delta p$$

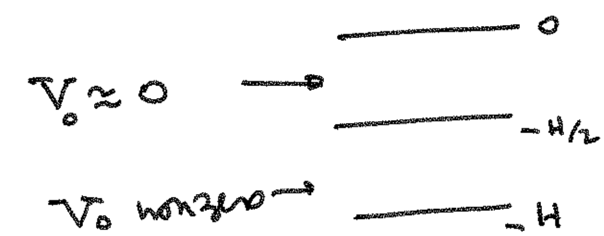
avg. for $z \in [-H, 0]$

δp is determined by the constraint

$$\Lambda = \frac{\partial}{\partial x}\delta v^x + \frac{\partial}{\partial y}\delta v^y = 0$$

by taking the divergence of these equations. This is a closed system, and convection of T does not enter.

e.) To develop a system of equations in which the fluid flow interacts with the convection of T, divide the fluid into two layers



and analyze the dynamics of the bottom layer using simplifying approximations. Let $V^z = \delta v^z(-H/2)$ be nonzero. Then, repeating the arguments of (d), ...

$$\int_{-H}^{-H/2} dz \left(\frac{\partial}{\partial x}\delta v^x + \frac{\partial}{\partial y}\delta v^y \right) = \frac{H}{2}\Lambda = -\int_{-H}^{-H/2} dz\, \frac{\partial}{\partial z}\delta v^z$$

$$= -\left[\delta v^z(-H/2) - \delta v^z(-H)\right]$$

or

$$\frac{H}{2}\Lambda = -V^z$$

f.) Now we get a system of equations in (x,y):

let V_0 = avg of $V_0(z)$ over $z = (-H/2, -H)$.

then

$$\frac{\partial}{\partial t} \delta v^x + V_0 \frac{\partial}{\partial x} \delta v^x = f \delta v^y - \frac{1}{\rho_0} \frac{\partial}{\partial x} \delta p$$

$$\frac{\partial}{\partial t} \delta v^y + V_0 \frac{\partial}{\partial x} \delta v^y = -f \delta v^x - \frac{1}{\rho_0} \frac{\partial}{\partial y} \delta p$$

$$\frac{\partial}{\partial x} \delta v^x + \frac{\partial}{\partial y} \delta v^y + V^z \cdot \frac{2}{H} = 0$$

$$\left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial x}\right) \delta T + \delta v^y A = 0$$

and a more complicated z equation that, in this exam, we replace by the more simple:

$$0 = g \rho \alpha \, \delta T \, \frac{H}{2} + \delta p$$

Now expand the five variables in Fourier modes in t, x, y:

e.g. $$\delta v^x = v^x \cdot e^{-i\omega t + i k^x x + i k^y y}$$

then gives

$$(-i\omega + ik^x V_0)\, V^x = f V^y - ik^x \frac{P}{\rho_0}$$

$$(-i\omega + ik^x V_0)\, V^y = -f V^x - ik^y \frac{P}{\rho_0}$$

$$\Lambda = +ik^x V^x + ik^y V^y = -\frac{2}{H} V^z$$

$$(-i\omega + ik^x V_0)\, T = -V^y A$$

$$P = -g\rho_0 \alpha \frac{H}{2} T$$

g.) Now

$$\Lambda = ik^x V^x + ik^y V^y$$
$$\Omega = ik^x V^y - ik^y V^x$$

$k^2 = (k^x)^2 + (k^y)^2$

$\downarrow$

$$(-i\omega + ik^x V_0)\, \Lambda = f\cdot(ik^x V^y - ik^y V^x) + k^2 \frac{P}{\rho_0}$$

$$(-i\omega + ik^x V_0)\, \Omega = f\cdot(-ik^x V^x - ik^y V^y)$$

so

$$-i(\omega - k^x V_0)\, \Lambda = \Omega f + k^2 \frac{P}{\rho_0}$$

$$-i(\omega - k^x V_0)\, \Omega = -\Lambda f$$

If we had $\Lambda = 0$ as in the previous setup, this system would be trivial. Instead, procede by eliminating P:

$$-i(\omega - k^x V_0)\,\Lambda = \Omega f - k^2 g\alpha \frac{H}{2} T$$

$$[-i(\omega - k^x V_0)]^2 \Lambda = -i(\omega - k^x V_0)\,\Omega f$$

$$- k^2 g\alpha \frac{H}{2} (-i\omega - k^x V_0) T$$

$$= -f^2 \Lambda + k^2 g\alpha \frac{H}{2} \cdot A v^y$$

so

$$(\omega - k^x V_0)^2 \Lambda = f^2 \Lambda - k^2 g\alpha \frac{H}{2} A v^y$$

now

$$v^y = \frac{-ik^y \Lambda - ik^x \Omega}{k^2}$$

$$= -i\frac{k^y \Lambda}{k^2} - i\frac{k^x}{k^2}\left(\frac{-if}{(\omega - k^x V_0)} \Lambda\right)$$

so

$$(\omega - k^x V_0)^2 \Lambda = f^2 \Lambda + i k^y g\alpha \frac{H}{2} A \Lambda$$

$$+ k^x \frac{f}{(\omega - k^x V_0)} g\alpha \frac{H}{2} A \Lambda$$

Cancelling Λ, this is a cubic equation for $\omega(k)$.

h) In the limit $k^y \gg k^x$ this equation is easy to analyze, since the last term is unimportant. Then

$$\omega = k^x V_0 \pm \left(f^2 + i k^y g\alpha \frac{H}{2} A \right)^{1/2}$$

in particular, the solution

$$\omega = k^x V_0 + f + i \frac{g\alpha H A}{4f} k^y$$

has $\mathrm{Im}\,\omega > 0$, an instability!