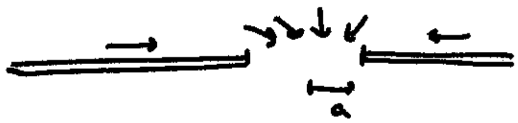
Physics 211 – Final Exam
Solutions

1.) a.) We want to find $\vec{v} = \vec{\nabla}\phi$ with $\nabla^2\phi = 0$ with the boundary condition



$v_\perp = -v$ in the hole only.

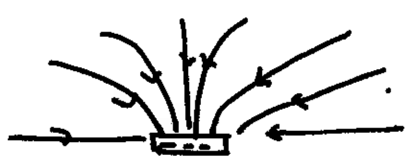
The analogue electrostatics problem is

σ charge = 0

charge density = constant such that

$$\int_{|r|<a} d^2s \; \hat{n}\cdot\vec{E} = -\pi a^2 v$$

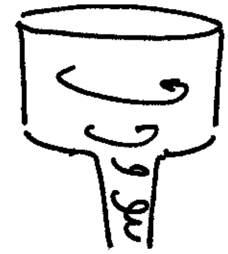
so $\vec{E}$ or $\vec{v}$ is the field of this charge distribution



for $|\vec{x}| \gg a$ this is a monopole

$$\vec{v} = -\frac{v}{2}\frac{a^2}{r^2}\hat{r}$$

b.) If $\eta = 0$, the only forces on a fluid element are pressure and gravity. Both are in the (r, z) plane. A torque requires a force $\parallel \hat{\phi}$.



Then $\Delta L_z = \rho\, r\, v_\phi\, \Delta \mathrm{Vol}$ remains fixed

$$v_\phi \sim \frac{1}{r} \qquad \text{or} \qquad v_\phi = V_0 \frac{a}{r}$$

The vorticity is

$$\vec{\omega} = \vec{\nabla} \times \vec{v} \simeq \hat{z}\, \frac{1}{r} \frac{\partial}{\partial r}(r v_\phi) = 0$$

But actually

$$\int d\vec{a} \cdot \vec{\omega} \cdot \hat{n} = \oint d\vec{l} \cdot \vec{v} = 2\pi V_0 a$$

so $\vec{\omega}$ is nonzero just at $\vec{r} = 0$: $\quad \vec{\omega} = \hat{z}\; 2\pi a V_0\, \delta^{(2)}(x_\perp)$

however, this is not relevant; see (c)

c.) From Bernoulli's theorem

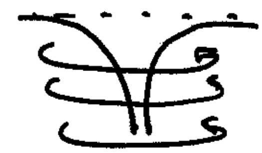
$$\frac{p}{\rho} + \frac{1}{2} v^2 + g z = 0$$

at the surface, $p = p_0$ atmospheric pressure. So

$$z = -\frac{1}{2g} v^2$$

or

$$z = -\frac{1}{2g}\frac{(V_0 a)^2}{r^2}$$



d.) In a rotating coordinate system, we add in the Coriolis force

$$\rho \frac{\partial \vec{v}}{\partial t} = -2\rho\, \vec{\Omega} \times \vec{v}$$

so, from above, if $\vec{\Omega} \parallel \hat{z}$

on the surface of the earth, in the northern hemisphere, we have effectively $\vec{\Omega} = +\hat{z}\left(\frac{2\pi}{\text{day}}\right) \cdot \cos\Theta$

The swirl is in the positive (counterclockwise) direction seen from above.

The effect is very weak $\Omega \cdot (\text{char time}) \sim \frac{\text{sec}}{\text{day}} \sim 10^{-5}$

e.) In cylindrical coordinates, the ϕ component of the N-S equation is

$$\frac{\partial v_\phi}{\partial t} + \vec{v}\cdot\vec{\nabla}\, v_\phi + \frac{v_r v_\phi}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \phi} + \nu\,(\nabla^2 \vec{v})_\phi$$

In this setup $v_r = 0$ $\quad$ p = funct of r, z only $\quad \frac{\partial}{\partial\phi} p = 0$

v_ϕ = funct of r, z only $\quad \frac{\partial}{\partial \phi} v_\phi = 0$

So

$$\frac{\partial v_\phi}{\partial t} = \nu\,(\nabla^2 \vec{v})_\phi$$

for a thin boundary layer near $r = a$ $\qquad \delta \ll a$

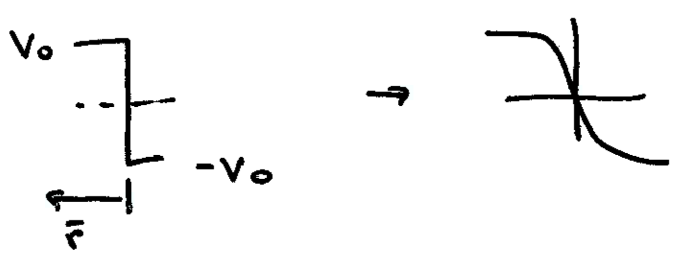
$$\frac{\partial v_\phi}{\partial t} = \nu \frac{d^2}{d\bar{r}^2} v_\phi \qquad \bar{r} = a - r$$

The boundary conditions are: $\quad v_\phi = V_o$ at $t = 0, \bar{r} > 0$

$v_\phi = 0$ at $\bar{r} = 0$

$v_\phi \to V_o$ as $\bar{r} \to \infty$

We can solve this by "method of images" for a diffusion problem



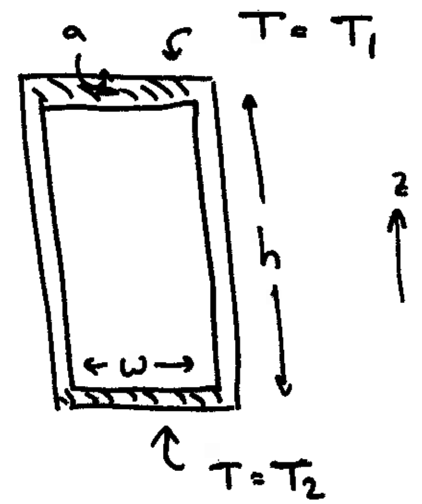
we discussed this diffusion problem in class. The solution is

$$V_\phi = V_0 \, \mathrm{erf}\left(\frac{r}{\sqrt{4\nu t}}\right) = \frac{2V_0}{\sqrt{\pi}} \int_0^{r/\sqrt{4\nu t}} dw \, e^{-w^2}$$

In particular

$$\delta = \sqrt{4\nu t}$$

2.)



a.) Poiseuille flow in the pipe gives a velocity distribution

$$v(r) = V_c\left(1 - r^2/a^2\right)$$

The force/area on the fluid is

$$\eta \left.\frac{\partial v}{\partial r}\right|_{r=a} = 2\eta \frac{V_c}{a}$$

The force/length $= 2\eta \dfrac{V_c}{a} \cdot 2\pi a = 4\pi \eta V_c$

The velocity averaged over the cross section is

$$V \cdot \pi a^2 = \int_0^a dr\, r\, 2\pi\, V_c \left(1 - r^2/a^2\right)$$

$$= 2\pi V_c \left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2} a^2 V_c$$

so

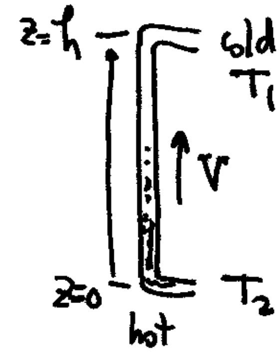
$$V = \tfrac{1}{2} V_c$$

The equation of motion is

$$\pi a^2 \rho \cdot \text{length} \cdot \frac{\partial V}{\partial t} = -4\pi \eta V_c l = -8\pi \eta V \cdot l$$

$$\frac{\partial V}{\partial t} = -\frac{8\nu}{a^2} V$$

b.) In the vertical pipes, T obeys



$$\frac{\partial T}{\partial t} + \underbrace{2V}_{V_c} \frac{\partial T}{\partial z} = \chi \frac{\partial^2}{\partial z^2} T$$

For $V = 0$, the steady state is

$$\frac{\partial^2}{\partial z^2} T = 0 \Rightarrow T = T_2 - \Delta T \frac{z}{h}$$

For $V > 0$, the steady state is the solution of

$$2V \frac{dT}{dz} = \chi \frac{d^2}{dz^2} T$$

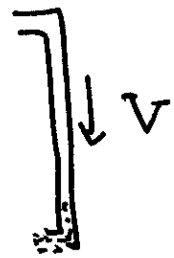
$$T = a + b \exp\left[\frac{2Vz}{\chi}\right]$$

We must have $T = T_2$ at $z = 0$ $\quad T = T_2 - \Delta T$ at $z = h$

so

$$T = T_2 - \Delta T\left(\frac{\exp\left[\frac{2Vz}{\chi}\right] - 1}{\exp\left[\frac{2Vh}{\chi}\right] - 1}\right)$$

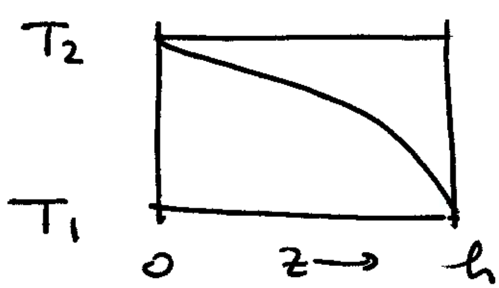
In the other pipe:



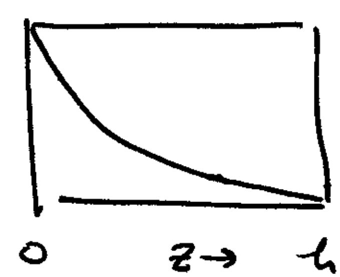
the positive direction for V
is down, so we send
$V \to -V$

$$T = T_2 - \Delta T\left(\frac{1 - \exp\left[-\frac{2Vz}{\chi}\right]}{1 - \exp\left[-\frac{2Vh}{\chi}\right]}\right)$$

going up:



going down:



c.) In the Boussinesq approximation, the bouyancy force per unit volume is

$$\rho \alpha T' g \qquad \underline{\text{upward}}$$

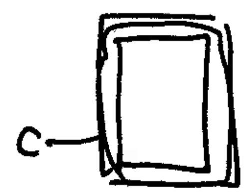
the force per unit length is

$$F/\ell = \pi a^2 \cdot \rho g \alpha T'$$

To feel more comfortable about this, note that if T were constant there would be no net force on the fluid. Note also that pressure exerts no net force on the fluid,

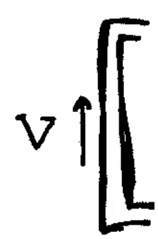
since

$$\oint_C dx \cdot \vec{\nabla} p = 0$$



Now, the upward force on the left is

$$\pi a^2 \rho g \alpha \int_0^h dz \; T(z)$$



$$= \pi a^2 \rho g \alpha \left[T_2 h - \Delta T \cdot \left(\frac{\chi}{2V}\right) \frac{(e^{\frac{2Vh}{\chi}} - 1) - h}{e^{2Vh/\chi} - 1} \right]$$

$$= \pi a^2 \rho g \alpha \left[T_2 h - \Delta T \left[\frac{\chi}{2V} - \frac{h}{e^{2Vh/\chi} - 1} \right] \right]$$

now

$$\frac{\chi}{2V} - \frac{h}{e^{2Vh/\chi}-1} = \frac{\chi}{2V} - \frac{h}{\left(\frac{2Vh}{\chi}\right) + \frac{1}{2}\left(\frac{2Vh}{\chi}\right)^2 + \frac{1}{3!}\left(\frac{2Vh}{\chi}\right)^3 + \cdots}$$

$$= \frac{\chi}{2V}\left[1 - \frac{1}{1 + \frac{1}{2}\left(\frac{2Vh}{\chi}\right) + \frac{1}{6}\left(\frac{2Vh}{\chi}\right)^2 + \cdots}\right]$$

$$= \frac{\chi}{2V}\left[1 - \left(1 - \frac{1}{2}\left(\frac{2Vh}{\chi}\right) + \frac{1}{4}\left(\frac{2Vh}{\chi}\right)^2 - \frac{1}{6}\left(\frac{2Vh}{\chi}\right)^2 + \cdots\right)\right]$$

$$= \frac{1}{2}h - \frac{1}{6}\frac{Vh^2}{\chi} + \cdots$$

so the upward force on the left is

$$\pi a^2 \rho g \alpha \left[(T_2 - \tfrac{1}{2}\Delta T)\, h + \frac{\Delta T}{6}\frac{V}{\chi}h^2 + \cdots\right]$$

The net force in the direction of V is (Left-Right)

$$\pi a^2 \rho g \alpha \frac{\Delta T}{3}\frac{V}{\chi}h^2$$

To add this to the equation on p. 6, note that this would come in the form:

$$\underbrace{\pi a^2 \rho\,(2h+2w)}_{\text{total mass}}\frac{dV}{dt} = \underbrace{\pi a^2 \rho g \alpha \frac{\Delta T}{3}\frac{V}{\chi}h^2}_{\text{net force}}$$

so $$\frac{dV}{dt} = +BV \qquad B = \frac{g\alpha\,\Delta T}{6\chi}\,\frac{h^2}{h+w}$$

d.) The complete equation is

$$\frac{dV}{dt} = -\frac{8\nu}{a^2} V + \frac{g\alpha}{6\chi} \Delta T \frac{h^2}{h+w} V + \dots$$

so V decreases exponentially when ΔT is small but increases exponentially when

$$\Delta T > \frac{48\nu}{a^2} \frac{\chi}{g\alpha} \frac{(h+w)}{h^2}$$

The sign of V is chosen randomly. This is spontaneous symmetry breaking.

e.) The full equation for V is

$$\frac{dV}{dt} = -\frac{8\nu}{a^2} V + \frac{\pi a^2 g \rho \alpha}{\pi a^2 \rho (2h+2w)} \cdot f$$

where

$$f = T_2 h - \Delta T \left[\frac{\chi}{2V} - \frac{h}{e^{2Vh/\chi} - 1} \right]$$

$$- \left(T_2 h - \Delta T \left[\frac{\chi}{2V} - \frac{h}{e^{-2Vh/\chi} - 1} \right] \right)$$

$$= \Delta T \left[\frac{h}{e^{2Vh/\chi} - 1} - \frac{h}{e^{-2Vh/\chi} - 1} - 2\frac{\chi}{2V} \right]$$

$$= \Delta T \left[\frac{h}{e^{2Vh/\chi}-1} + \frac{h\, e^{2Vh/\chi}}{e^{2Vh/\chi}-1} - 2\,\frac{\chi}{2V} \right]$$

$$= \Delta T \left[h\, \frac{e^{2Vh/\chi}+1}{e^{2Vh/\chi}-1} - \frac{\chi}{V} \right]$$

$$= \Delta T \left[h \coth \frac{Vh}{\chi} - \frac{\chi}{V} \right]$$

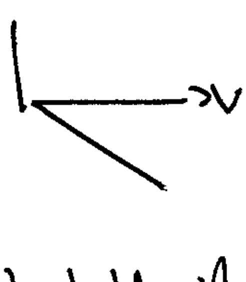
Check:

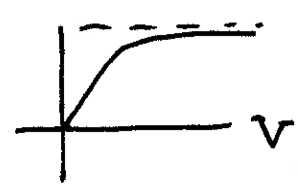
$$f = \Delta T \cdot h \cdot \left[\frac{\chi}{Vh} + \frac{1}{3}\frac{Vh}{\chi} + \cdots - \frac{\chi}{Vh} \right]$$

$$= \Delta T \cdot \frac{1}{3} \frac{Vh^2}{\chi} + \cdots \quad \checkmark$$

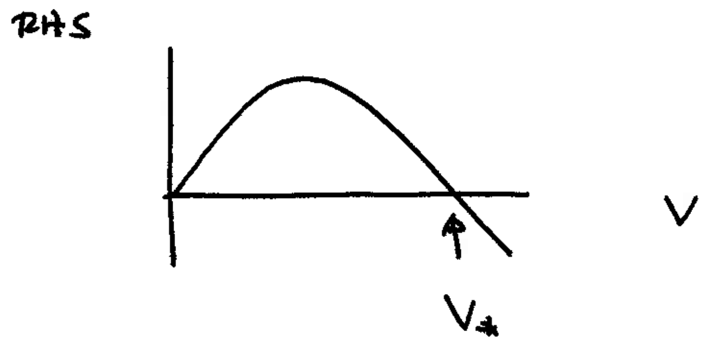
so

$$\frac{dV}{dt} = \underbrace{-\frac{8\nu}{a^2} V}_{\text{linear in } V} + \frac{g\alpha h}{2(h+w)} \Delta T \left[\underbrace{\coth \frac{Vh}{\chi} - \frac{\chi}{Vh}}_{\to 1 \text{ as } V \to \infty} \right]$$





Above threshold, the right-hand side has the form:



so V increases with time and asymptotically approaches V_* (constant velocity).

For ΔT very large, the right-hand side is approximately

$$-\frac{8\nu}{a^2} V + \frac{g\alpha h}{2(h+w)} \Delta T \cdot 1$$

and so
$$V_* = \frac{g\alpha a^2 h}{16\nu(h+w)} \Delta T$$

for large ΔT.