# Physics 211 — Problem Set #7

## Solutions

1.) a.)

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r} r v_r + \frac{1}{r}\frac{\partial}{\partial \phi} v_\phi$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r v_r$$

$$\vec{\nabla}\cdot\vec{v} = \begin{cases} -Q & r<L \\ 0 & r>0 \end{cases}$$

with $v_r$ continuous at $r=L$, then

$$v_r = \begin{cases} -\frac{Q}{2} r & r<L \\ -\frac{QL^2}{2r} & r>L \end{cases}$$

b.) Under the assumptions given, the N-S equation becomes

$$0 = -\vec{f}\times\vec{v} - \vec{\nabla}\varphi - a\vec{v}$$

$$\vec{f}\times\vec{v} = \underbrace{2\Omega\cos\theta}_{f_0}\left[-v_\phi\hat{r} + v_r\hat{\phi}\right]$$

take the curl of this equation

$$\vec{\nabla} \times \vec{v} = \hat{z} \; \frac{1}{r} \frac{\partial}{\partial r}(r\, v_\phi) + \cdots$$

$$\vec{\nabla} \times (-\vec{f} \times \vec{v}) = -f_o \frac{1}{r} \frac{\partial}{\partial r}(r\, v_r) = \begin{cases} f_o Q & r < L \\ 0 & r > 0 \end{cases}$$

then

$$0 = \nabla \times (-\vec{f} \times \vec{v}) - a\, \vec{\nabla} \times \vec{v}$$

$$0 = \hat{z} \begin{cases} f_o Q \\ 0 \end{cases} - a \frac{1}{r} \frac{\partial}{\partial r}(r\, \mathbf{V}_\phi)$$

so

$$v_\phi = \begin{cases} \dfrac{f_o Q}{2a}\, r & r < L \\[2ex] \dfrac{f_o Q}{2a} \dfrac{L^2}{r} & r > L \end{cases}$$

c.) It should now be that the $\hat{\phi}$ component of N-S is automatically zero, while the $\hat{r}$ component determines $\varphi(r)$

$$-\vec{f} \times \vec{v} - a \vec{v} = \hat{\phi} \begin{cases} +f_o \dfrac{Q}{2} r - a \dfrac{f_o Q}{2a} r \\[2ex] +f_o \dfrac{Q L^2}{2r} - a \dfrac{f_o Q}{2a} \dfrac{L^2}{r} \end{cases} = 0$$

$$+ \ \hat{r} \begin{cases} f_0 \dfrac{f_0 Q}{2a} r & - \ a\left(-\dfrac{Q}{2} r\right) \\ f_0 \dfrac{f_0 Q}{2a} \dfrac{r_1^2}{r} & - \ a\left(-\dfrac{Q r_1^2}{2r}\right) \end{cases}$$

so

$$-\hat{f} \times \vec{v} - a\vec{v} = \hat{r} \begin{cases} \dfrac{Q r}{2} \cdot \left(\dfrac{f_0^2}{a} + a\right) \\ \dfrac{Q r_1^2}{2r} \cdot \left(\dfrac{f_0^2}{a} + a\right) \end{cases}$$

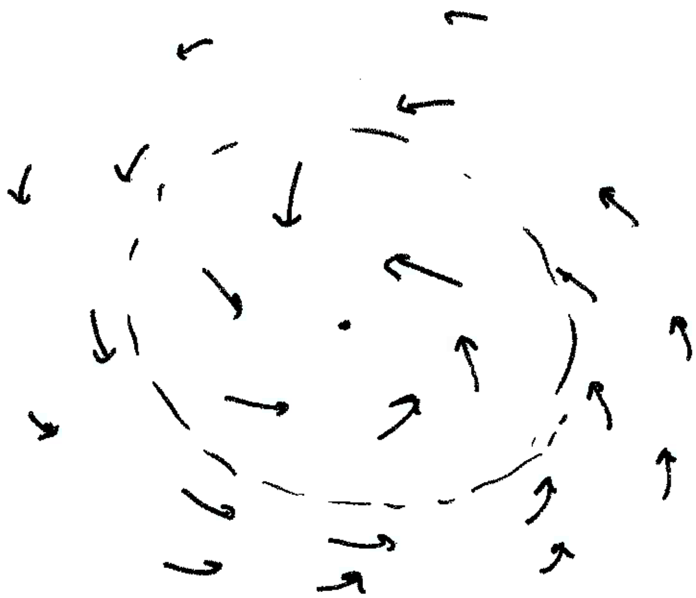
$$= \vec{\nabla} \varphi$$

so $\vec{\nabla}\varphi$ = positive $\cdot \ \hat{r}$

is <u>low</u> pressure in the center

$$\varphi = \begin{cases} \varphi_0 + \dfrac{Q r^2}{4}\left(\dfrac{f_0^2}{a} + a\right) & r < r_1 \\ \varphi_0 + \dfrac{Q r_1^2}{4}\left(\dfrac{f_0^2}{a} + a\right) + \dfrac{Q r_1^2}{2}\left(\dfrac{f_0^2}{a} + a\right) \log \dfrac{r}{r_1} & r > r_1 \end{cases}$$

The pattern is



with low pressure in the center  $\vec{\nabla}\times\vec{v} = 0$ outside the center

2.) a.) The surface winds set up an Ekman layer at the surface of the ocean. The depth of this layer should be

$$d \sim \left(\frac{\nu}{\Omega}\right)^{1/2}$$

Putting in $\nu = 10^{-2}\ \text{cm}^2/\text{sec}$ $\quad \Omega = 2\pi/\text{day} = 7\times10^{-5}/\text{sec}$,

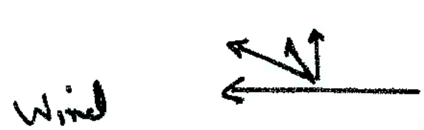
we find

$$d \sim 12\ \text{cm}.$$

This does not make sense, being smaller than the typical height of an ocean wave. If we assume turbulence and thus a larger effective viscosity in the top layer of the ocean, $d$ will be larger. For $d \sim 30\text{m}$, we need

$$\nu_{eddy} \sim 6\times10^{-2}\ \text{m}^2/\text{sec} \sim 6\times10^{2}\ \text{cm}^2/\text{sec}$$

b.) In the Ekman layers set up in the trade winds and westerly winds regions, the current turns toward the Sargasso Sea as $z$ decreases:

The ocean must reach an equilibrium in which this flow is resisted by a pressure gradient that gives higher pressure in the Sargasso Sea and lower pressure on the wind paths. To balance pressure against the Coriolis force, we need

$$|\vec{\nabla} p| \sim \rho \; |2\vec{\Omega}\times\vec{v}\,|$$

$$\sim \rho_0 \cdot (1.4\times10^{-5}/\text{sec}) \cdot (\underbrace{10\,\text{m/sec}}_{\text{typical speed of wind of currents}})$$

$$\sim \rho_0 \cdot 1.4\times10^{-4}\,\text{m/sec}^2$$

the extra pressure needed is then

$$\Delta p = \rho_0\, g\, \Delta z \sim W \rho_0 \,(1.4\times10^{-4}\,\text{m/sec}^2)$$

$W =$ width of region of pressure drop.

$\Delta z = 1.5$ m $\qquad g = 10$ m/sec$^2$ $\qquad$ so

$$\Delta z \sim W \cdot \frac{1.4\times10^{-4}\,\text{m/sec}^2}{10\,\text{m/sec}^2}$$

which is reasonable with $\Delta z = 1.5$ m for

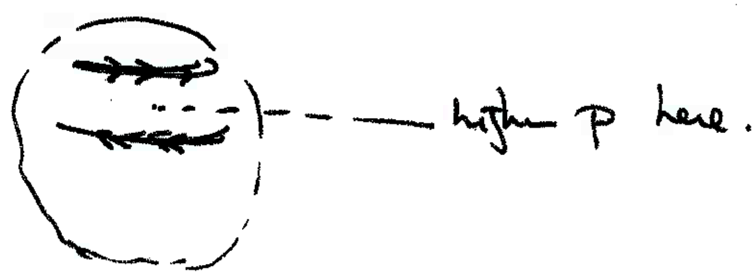
$W \sim 100$ km

c.) At lower depths, the Ekman layer is no longer present, but the pressure gradient is still there. The pressure gradient now drives a flow around the Sargasso sea according to

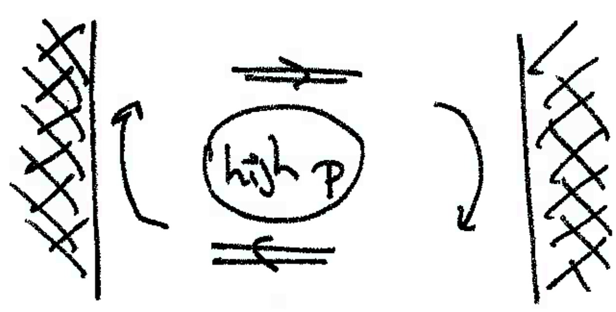
$$-\frac{1}{\rho}\vec{\nabla}p - 2\vec{\Omega}\times\vec{v} = 0$$

Reversing the above computation, we expect a flow at lower depths that is also ~ 10 m/sec.

d.) If there were no continents, we would expect global flows




However, the continental land masses impede these flows. Instead, we have




The north-south currents are also driven by pressure against Coriolis.

3.) a.) The fluid is incompressible, so

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0$$

$v_z = 0$ at the bottom, $z=0$; at the top, $z=h$

$$v_z = \frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x}$$


Integrate the first equation from $z=0$ to $z=h$:

$$h \frac{\partial v_x}{\partial x} + \int_0^h dz \frac{\partial v_z}{\partial z} = 0$$

$$h \frac{\partial v_x}{\partial x} + v_z(h) - v_z(0) = 0$$

$$h \frac{\partial v_x}{\partial x} + \frac{\partial h}{\partial t} - v_x \frac{\partial h}{\partial x} = 0$$

then

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h v_x) = 0$$

The pressure is $p_0$ at $z=h$; at a general $z$

$$p = p_0 + \rho g (h - z)$$

then
$$\frac{\partial P}{\partial x} = \rho g \frac{\partial h}{\partial x}$$

the x-component of the Navier-Stokes equation is

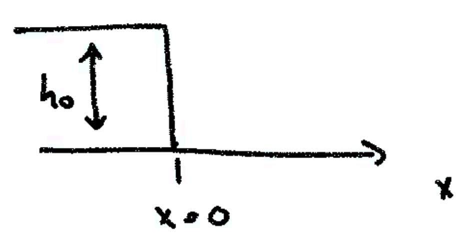
$$\frac{\partial V_x}{\partial t} + \underbrace{\vec{V}\cdot\vec{\nabla}\, V_x}_{\approx\ V_x \frac{\partial}{\partial x} V_x} = -\frac{1}{\rho}\frac{\partial}{\partial x} P = \underbrace{-g\frac{\partial h}{\partial x}}_{\text{from above}}$$

then if we set $V_x = v$

① $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv) = 0$$

② $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

b.) The initial condition is




Let $v_0 = (g h_0)^{\frac{1}{2}}$. Look for a solution

$$h = h_0 \bar{h}(\xi) \qquad v = v_0 \bar{v}(\xi) \qquad \xi = \frac{x/t}{v_0}$$

eq ① $\Rightarrow$ $$h_0\left(-\frac{\xi}{t}\right)\bar{h}' + h_0 v_0 \frac{1}{t v_0}\left(\bar{h}'\bar{v} + \bar{h}\bar{v}'\right) = 0$$

or $$(\bar{v}-\xi)\,\bar{h}' + \bar{h}\bar{v}' = 0$$

eq. ② $\Rightarrow$ $$v_0\left(-\frac{\xi}{t}\right)\bar{v}' + v_0^2\frac{1}{t v_0}\bar{v}\bar{v}' + \frac{g h_0}{v_0 t}\bar{h}' = 0$$

or $$(v-\xi)\,\bar{v}' + \bar{h}' = 0$$

then $$(\bar{v}-\xi)^2\,\bar{h}' = -(\bar{v}-\xi)\,\bar{h}\bar{v}' = \bar{h}\,\bar{h}'$$

thus $$(\bar{v}-\xi)^2 = \bar{h} \qquad \bar{v} = \xi \pm \sqrt{\bar{h}}$$

also $$\bar{v}' = -\frac{\bar{h}'}{(\bar{v}-\xi)} = \mp\frac{\bar{h}'}{\sqrt{\bar{h}}} = \frac{d}{d\xi}\left(\mp 2\sqrt{\bar{h}}\right)$$

so

$$\bar{v} = \xi \pm \sqrt{\bar{h}} = (\text{const}) \mp 2\sqrt{\bar{h}}$$

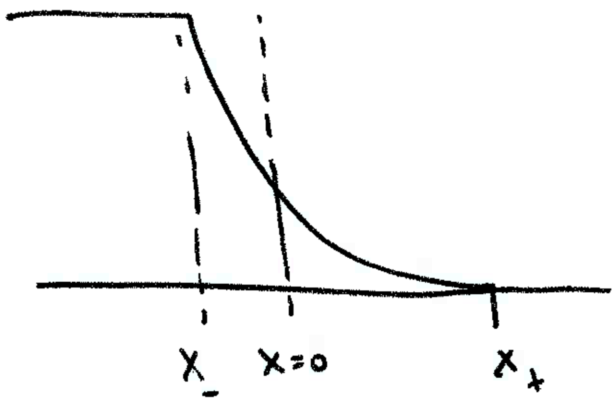
$$\sqrt{\bar{h}} = \pm\frac{1}{3}(\xi - c)$$

$$\bar{h} = \frac{1}{9}\left(c - \frac{x}{v_0 t}\right)^2$$

Note that $\bar{h}(x=0) = \dfrac{c^2}{9}$ = const. as a funct of $t$

$$h = h_0\frac{c^2}{9}\left(1 - \frac{x}{c v_0 t}\right)^2$$

This solution has the form of a parabola:




at $x_+ = cv_0 t$, $h=0$. Then $h=0$ at all $x$ to the right of $x_+$

at $x_- = -cv_0 t\left(\frac{3}{c}-1\right)$, $h=h_0$. Then $h=h_0$ at all $x$ to the left of $x_-$

We still need to determine $C$. This is done by conservation of water:

(original amt of water to the right of $x_-$) $= h_0 |x_-|$

$$= \int_{x_-}^{x_+} dx \; h(x,t) \qquad y = 1-\frac{x}{cv_0 t}$$

$$= \int_{x_-}^{x_+} dx \; h_0 \frac{c^2}{9}\left(1-\frac{x}{cv_0 t}\right)^2 \qquad dy = -\frac{dx}{cv_0 t}$$

$$= h_0 \frac{c^2}{9}(cv_0 t)\int_0^{3/c} dy\; y^2 = h_0 \frac{c^3}{9} v_0 t \, \frac{1}{3}\left(\frac{3}{c}\right)^3$$

so

$$h_0 |x_-| = h_0 v_0 t$$

$$c v_0 t \left(\frac{3}{c} - 1\right) = v_0 t$$

$$3 - c = 1 \quad \Rightarrow \quad c = 2$$

then

$$h = h_0 \frac{4}{9}\left(1 - \frac{x}{2v_0 t}\right)^2$$

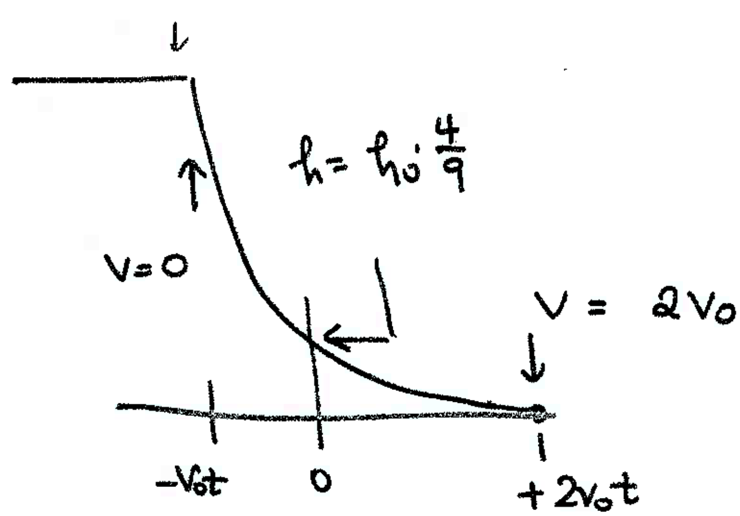
$$v = v_0 \left(\xi + \sqrt{\bar{h}}\right)$$

$$= v_0 \left[\frac{x}{v_0 t} + \frac{2}{3}\left(1 - \frac{x}{2v_0 t}\right)\right]$$

$$= v_0 \left[\frac{2}{3} + \frac{2}{3}\frac{x}{v_0 t}\right]$$

$$v = v_0 \cdot \frac{2}{3}\left(1 + \frac{x}{v_0 t}\right)$$

then

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The point at $x=0$ stays fixed, and otherwise the shape gets wider with time $\propto$ $V_0 t$.

c.) Return to

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial x} + h\frac{\partial v}{\partial x} = 0$$

multiply the second equation by $\pm \frac{g}{\sqrt{gh}}$ and add it to the first

$$\left(\frac{\partial v}{\partial t} \pm \frac{1}{\sqrt{gh}}\frac{\partial}{\partial t}(gh)\right) + \left(v \pm \sqrt{gh}\right)\left(\frac{\partial v}{\partial x} \pm \frac{1}{\sqrt{gh}}\frac{\partial}{\partial x}(gh)\right)$$

$$= 0$$

or

$$\frac{\partial}{\partial t}\left(v \pm 2\sqrt{gh}\right) + \left(v \pm \sqrt{gh}\right)\frac{\partial}{\partial x}\left(v \pm 2\sqrt{gh}\right)$$

$$= 0$$

let

$$J_\pm = v \pm 2\sqrt{gh}$$

$$V_\pm = (v \pm \sqrt{gh})$$

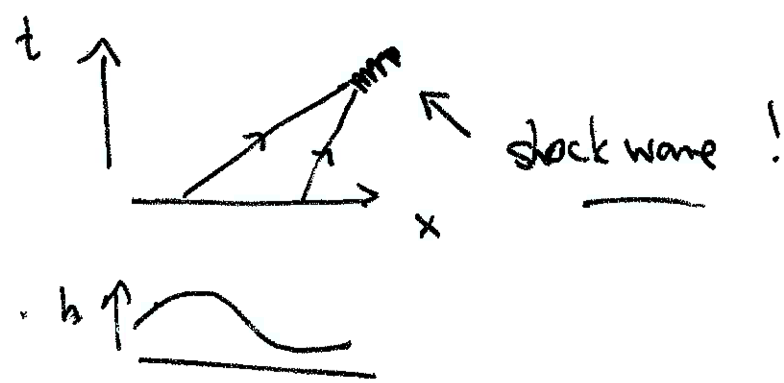
$J_\pm$ are Riemann invariants satisfying

$$\left( \frac{\partial}{\partial t} + v_\pm \frac{\partial}{\partial x} \right) J_\pm = 0$$

d.) The characteristics satisfy

$$\frac{d}{dt} X = v_\pm$$

For right-moving disturbances $v_+ = v + \sqrt{gh}$ increases with $h$. So characteristics in regions of high $h$ catch up to those in regions of low $h$



e.) Looking back at p. 5, we see

$$J_+ = v + 2\sqrt{gh}$$

$$= V_0\left(\frac{2}{3} + \frac{2}{3}\frac{x}{V_0 t}\right) + \frac{4}{3}V_0 - \frac{2}{3}\frac{V_0 x}{V_0 t}$$

$$= 2V_0 = (\underline{\text{const}})$$

so look for a solution with $J_+ = \text{constant}$.

$$\frac{\partial}{\partial t} J_- + v_- \frac{\partial}{\partial x} J_- = 0$$

$$v_- = \frac{3J_- + J_+}{4}$$

look for a solution of the form $J_- = A\frac{x}{t} + B$

$$-A\frac{x}{t^2} + \left[\frac{3}{4}\left(A\frac{x}{t} + B\right) + \frac{1}{4}J_+\right]\frac{A}{t} = 0$$

then $-A + \frac{3}{4}A^2 = 0 \Rightarrow A = \frac{4}{3}$

$$\frac{3}{4}B + \frac{1}{4}J_+ = 0 \Rightarrow B = -\frac{1}{3}J_+$$

for $J_+ = 2V_0$ this gives

$$J_- = \frac{4}{3}\frac{x}{t} - \frac{2}{3}V_0$$

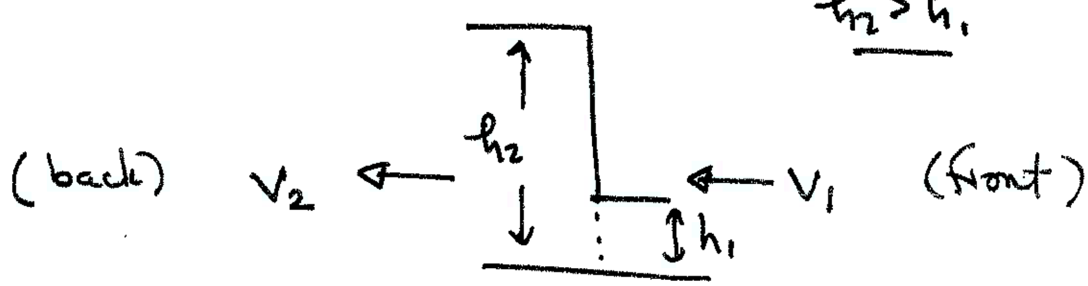
and, from p. 5

$$V - 2\sqrt{gh} = V_0\left(\frac{2}{3} + \frac{2}{3}\frac{x}{V_0 t}\right) - \frac{4}{3}V_0 + \frac{2}{3}\frac{x}{t}$$

$$= \frac{4}{3}\frac{x}{t} - \frac{2}{3}V_0 \qquad \checkmark$$

4.) a) Go to the frame in which the jump discontinuity is at rest. In that frame, we have

$\underline{h_2 > h_1}$




The mass flow from right to left is

$$\int_0^h dz\, j^x = \rho h_1 V_1 = \rho h_2 V_2$$

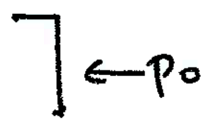
The momentum flow across the jump is

$$\int_0^h dz\; T^{xx} = \int_0^h dz\; (p + \rho v^2)$$

$$= p_0 h + \int_0^h dz\; \rho g (h - z) + h \rho v^2$$

$$= p_0 h + \frac{1}{2} \rho g h^2 + h \rho v^2$$

There is also an air pressure term




so actually the first term is $p_0 h_2$ on both sides and has no discontinuity. Then the jump condition is

$$\rho h_1 v_1^2 + \frac{1}{2} \rho g h_1^2 = \rho h_2 v_2^2 + \frac{1}{2} \rho g h_2^2$$

b.) We can solve these conditions for $v_1 v_2$ in terms of $h_1, h_2$:

$$v_2 = \frac{h_1}{h_2} v_1$$

$$h_1 v_1^2 - h_2 v_2^2 = \frac{1}{2} g (h_2^2 - h_1^2)$$

$$h_1 v_1^2 \left(1 - \frac{h_1}{h_2}\right) = \frac{1}{2} g \left(h_2^2 - h_1^2\right)$$

$$\frac{h_1}{h_2} v_1^2 \left(h_2 - h_1\right) = \frac{1}{2} g \left(h_2^2 - h_1^2\right)$$

$$v_1^2 = \frac{1}{2} g \left(h_2 + h_1\right)\left(\frac{h_2}{h_1}\right)$$

$$v_1 = \left[ g \frac{(h_1+h_2)}{2} \frac{h_2}{h_1} \right]^{1/2}$$

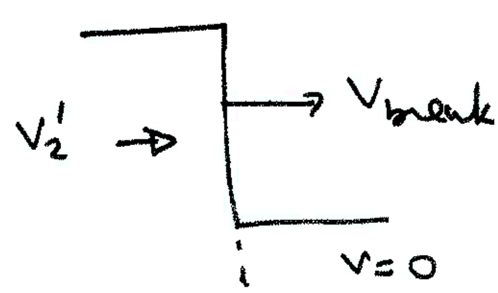
$$v_2 = \left[ g \frac{h_1+h_2}{2} \frac{h_1}{h_2} \right]^{1/2}$$

so

$$v_1 > \sqrt{g h_1} \qquad\qquad v_2 < \sqrt{g h_2}$$

supersonic          subsonic

c.) Now boost to the frame where the water to the right of the break is at rest.



$$V_{break} = V_1 = \left[ \frac{g(h_1+h_2)}{2} \frac{h_2}{h_1} \right]^{1/2}$$

$$V_2' = V_1 - V_2 = V_{break} \cdot \left(1 - \frac{h_1}{h_2}\right)$$

$$= \left[ \frac{g}{2} \frac{h_2^2 - h_1^2}{h_2^2} (h_2 - h_1) \right]^{1/2}$$