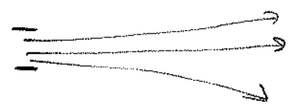
Physics 211 - Problem Set #6

Solutions

1.) a.)

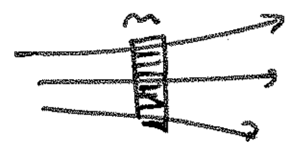


The fluid momentum density is

$$P_x = \rho v_x$$

At position x, the fluid emitted in an interval Δt occupies a slice of width

$$\Delta x = v_x \, \Delta t$$



The momentum in this slice is

$$\int dx\,dy \; \rho v_x = \int_{-\infty}^{\infty} dy \; \rho v_x^2 \, \Delta t = F \, \Delta t$$

per unit length in z. Viscosity transfers momentum laterally, from one value of y to another, so this quantity is conserved in the flow.

b.) In terms of the stream function ψ

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

so

$$v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x = \nu \frac{\partial^2}{\partial y^2} v_x$$

becomes

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

subscript = derivative

Look for a scaling solution

$$\psi = A x^p f(\eta) \qquad \eta = \frac{y}{x^q}$$

$$\psi_y = A x^{p-q} f'(\eta) \qquad \psi_x = A\left(p x^{p-1} f - q x^{p-1} \frac{y}{x^q} f'\right)$$

$$= A x^{p-1}(pf - q\eta f')$$

so

$$\psi_y \psi_{xy} \sim x^{2p-2q-1}$$

$$\psi_x \psi_{yy} \sim x^{2p-2q-1}$$

$$\psi_{yyy} \sim x^{p-3q}$$

so for a scaling solution $\quad 2p-2q-1 = p-3q$

or $\quad p+q=1$

We need one more equation between p and q

This comes from the constancy of F:

$$F/\rho = \int dy\, v_x^2 = \int dy\, (\psi_y)^2 = \int d\eta\; x^q\, x^{2p-2q} \left(A f'(\eta)\right)^2$$

so $\quad 2p - q = 0$

so $\quad p = \frac{1}{3} \quad q = \frac{2}{3}$

e.) Then $\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$ becomes

$$(A x^{-1/3} f')\; A x^{-2/3-2/3} \left(\tfrac{1}{3} f' - \tfrac{2}{3} f' - \tfrac{2}{3} \eta f''\right)$$

$$- A x^{-2/3} \left(\tfrac{1}{3} f - \tfrac{2}{3} \eta f'\right) A x^{-1/3-2/3} f''$$

$$- \nu A x^{\frac{1}{3}-2} f''' = 0$$

or

$$A^2 \left[-\tfrac{1}{3} (f')^2 - \cancel{\tfrac{2}{3} \eta f' f''} - \tfrac{1}{3} f f'' + \cancel{\tfrac{2}{3} \eta f' f''} \right]$$

$$- A\nu f''' = 0$$

set $A = 6\nu$ to simplify this

$$f''' + 2 f f'' + 2 (f')^2 = 0$$

d.) Try

$$f = B \tanh \alpha\eta$$

$$f' = B\alpha \frac{1}{\cosh^2 \alpha\eta}$$

$$f'' = -2B\alpha^2 \frac{\sinh \alpha\eta}{\cosh^3 \alpha\eta}$$

$$f''' = -2B\alpha^3 \left[\frac{\cosh \alpha\eta}{\cosh^3 \alpha\eta} - 3 \frac{\sinh^2 \alpha\eta}{\cosh^4 \alpha\eta} \right]$$

$$= -2B\alpha^3 \left[\frac{\cosh^2 \alpha\eta - \sinh^2 \alpha\eta}{\cosh^4 \alpha\eta} - 2 \frac{\sinh^2 \alpha\eta}{\cosh^4 \alpha\eta} \right]$$

$$f f'' + (f')^2 = + B^2\alpha^2 \left[\frac{1}{\cosh^4 \alpha\eta} - 2 \frac{\sinh^2 \alpha\eta}{\cosh^4 \alpha\eta} \right]$$

so

$$f''' = (-2) \left[f f'' + (f')^2 \right] \quad \text{if} \quad B = \alpha$$

so

$$f = \alpha \tanh \alpha\eta \quad \text{for some } \alpha.$$

e.)

$$F/\rho = \int_{-\infty}^{\infty} dy \, v_x^2 = A^2 \int_{-\infty}^{\infty} d\eta \left(\frac{\alpha^2}{\cosh^2 \alpha\eta} \right)^2$$

$$= 36\nu^2 \alpha^3 \int_{-\infty}^{\infty} d\omega \, \frac{1}{\cosh^4 \omega}$$

Now we just need to evaluate the integral.

$$\cosh w = \tfrac{1}{2}(e^{w} + e^{-w}) = \tfrac{1}{2} e^{-w} (e^{2w} + 1)$$

$$I = \int_{-\infty}^{\infty} dw \, \frac{1}{\cosh^4 w} = \int dw \; 16 \, e^{4w} \frac{1}{(e^{2w}+1)^4}$$

let $\quad z = \dfrac{e^{2w}}{e^{2w}+1} \qquad (1-z) = \dfrac{1}{e^{2w}+1}$

$$-dz = -2\,dw \, \frac{e^{2w}}{(e^{2w}+1)^2} = -2\,dw \; z(1-z)$$

$$I = 8 \int_0^1 \frac{dz}{z(1-z)} \, z^2(1-z)^2 = 8 \int_0^1 dz \; z(1-z) = \frac{8}{6} = \frac{4}{3}$$

so $\quad F/\rho = 48 \, \nu^2 \alpha^3$

$$\alpha = \left[\frac{F}{48 \nu^2 \rho} \right]^{1/3} \qquad \alpha\eta = \left[\frac{F}{48 \rho \nu^2 x^2} \right]^{1/3} y$$

check units $\quad \dfrac{F}{\rho} = \dfrac{[\text{gcm/sec}]/\text{cm sec}}{\text{g}/\text{cm}^3} \sim \text{cm}^3/\text{sec}^2$

$$\nu x = \text{cm}^2/\text{sec} \cdot \text{cm} \sim \text{cm}^3/\text{sec}$$

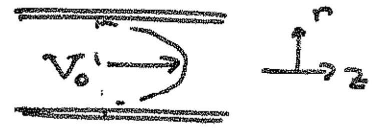
so $\quad \left[\dfrac{F}{\rho (\nu x)^2} \right]^{1/3} \sim \left[\dfrac{\text{cm}^3/\text{sec}^2}{\text{cm}^6/\text{sec}^2} \right]^{1/3} \sim \dfrac{1}{\text{cm}} \quad \checkmark$

finally!

$$V_x = \left[\frac{3}{24} \frac{F^2}{\rho^2 \nu^4} \frac{1}{x} \right]^{1/3} \frac{1}{\cosh^2 \left[\left[\frac{F}{48 \rho \nu^2 x^2} \right]^{1/3} y \right]}$$

2.) a.) In the pipe

$$v_z = V_0 \left(1 - r^2/a^2\right)$$



the source term for heating is

$$\frac{\nu}{2c_p}\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right)^2 = 2\,\frac{\nu}{2c_p}\left(\frac{\partial v_z}{\partial r}\right)^2$$

$(\sigma^{rz})^2$ and $(\sigma^{zr})^2$ contribute

$$= \frac{\nu}{c_p}\left(-2V_0\frac{r}{a^2}\right)^2 = \frac{4\nu V_0^2}{c_p}\,\frac{r^2}{a^4}$$

b.) In the steady state, T is a fnct of r only

so $\dfrac{\partial T}{\partial t} = 0 \qquad \vec{v}\cdot\vec{\nabla}\,T = 0$

then

$$0 = \chi\,\nabla^2 T + \frac{4\nu V_0^2}{c_p}\,\frac{r^2}{a^4}$$

$$\frac{1}{r}\frac{d}{dr}\,r\frac{d}{dr}\,T(r) = -\,4\nu\,\frac{V_0^2}{\chi c_p}\,\frac{r^2}{a^4}$$

$$r\frac{d}{dr}\,T(r) = A - \frac{\nu V_0^2}{\chi c_p}\,\frac{r^4}{a^4}$$

$A \neq 0 \Rightarrow T(r) \sim \log r$ as $r \to 0$ so $A = 0$

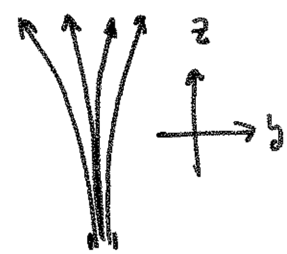
then

$$T(r) = C - \frac{\nu V_0^2}{4\chi C_p} \frac{r^4}{a^4}$$

$T(r) = T_0$ at $r = a$ so

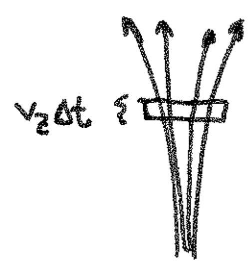
$$T(r) = T_0 + \frac{\nu V_0^2}{4\chi C_p}\left(1 - \frac{r^4}{a^4}\right)$$

3.) a.) In this problem, heat diffuses laterally as the fluid elements rise. The heat in each horizontal slice is constant.



The density of heat is $\rho C_p \cdot \Delta T$

The heat in a slice emitted in Δt per unit length is:



$$Q \Delta t = \int dy\,dz \; \rho C_p \Delta T$$

$$= \rho C_p \left(\int_{-\infty}^{\infty} dy \; V_z \Delta T \right) \Delta t$$

$$Q = \rho C_p \int_{-\infty}^{\infty} dy \; V_z \; \Delta T$$

b.) Under the assumptions of the problem

$$P = P_0 - \rho_0 g z$$

and in the plume $\rho = \rho_0 (1 - \alpha \Delta T)$

so there is a buoyancy force / cm^3

$$f = \alpha \rho_0 g \, \Delta T \quad \text{upward}$$

$$\rho \frac{D V_z}{Dt} = \rho V_z \frac{\partial}{\partial z} V_z = \alpha \rho_0 g \, \Delta T$$

then $$\frac{\partial}{\partial z} \left(\tfrac{1}{2} V_z^2\right) = \alpha g \, \Delta T$$

c.) $$\frac{D}{Dt} \delta^2 = V_z \frac{\partial}{\partial z} \delta^2 = \chi$$

Now assume

$$V_z = A z^\alpha \qquad \Delta T = B z^\beta \qquad \delta = C z^\gamma$$

Determine $\alpha \beta \gamma$:

$$\delta \cdot V_z \cdot \Delta T \sim 1 \quad \Rightarrow \quad z^{\alpha+\beta+\gamma} \sim 1$$

$$\frac{\partial}{\partial z} (V_z^2) \sim \Delta T \quad \Rightarrow \quad z^{2\alpha - 1} \sim z^\beta$$

$$V_z \frac{\partial}{\partial z} \delta^2 \sim 1 \quad \Rightarrow \quad z^{\alpha + 2\gamma - 1} \sim 1$$

so
$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \alpha \quad + 2\gamma &= 1 \\ 2\alpha-\beta \quad &= 1 \end{aligned}\right\} \rightarrow \quad \alpha = \frac{1}{5} \quad \beta = -\frac{3}{5} \quad \gamma = \frac{2}{5}$$

so $\delta \sim z^{2/5}$

d.) Finally, determine A, B, C

$$\frac{Q}{\rho c_p} = ABC$$

$$\frac{\partial}{\partial z}\left(\tfrac{1}{2}V_z^2\right) = A^2 z^{-3/5} = \alpha g\, B\, z^{-3/5}$$

$$V_z \frac{\partial}{\partial z}\delta^2 = AC^2 = \chi$$

$$ABC = A^{5/2}\frac{\chi^{1/2}}{\alpha g} = \frac{Q}{\rho c_p}$$

so $A = \left(\frac{Q}{\rho c_p}\cdot \alpha g\right)^{2/5}\chi^{-1/5} \qquad B = \left(\frac{Q}{\rho c_p}\right)^{4/5}(\alpha g)^{-1/5}\chi^{-2/5}$

$$C = \left(\frac{Q}{\rho c_p}\right)^{-1/5}(\alpha g)^{-1/5}\chi^{3/5}$$

thus:
$$V_z = \left(\frac{Q}{\rho c_p}\right)^{2/5}(\alpha g)^{2/5}\chi^{-1/5}\, z^{1/5}$$

$$\Delta T = \left(\frac{Q}{\rho c_p}\right)^{4/5}(\alpha g)^{-1/5}\chi^{-1/5}\, z^{-3/5}$$

$$\delta = \left(\frac{Q}{\rho c_p}\right)^{-1/5}(\alpha g)^{-1/5}\chi^{3/5}\, z^{2/5}$$

e.) Now redo the analysis for a point source

$$Q = \text{heat/sec} = \rho c_p \int d^2x_\perp \, V_z \, \Delta T \sim \rho c_p \, \delta^2 V_z \, \Delta T$$

and, still:

$$\frac{\partial}{\partial z}\left(\tfrac{1}{2} V_z^2\right) = \alpha g \, \Delta T$$

$$V_z \frac{\partial}{\partial z} \delta^2 = \chi$$

so if we assume the power law forms on p. 8

$$\left.\begin{aligned} \alpha + \beta + 2\gamma &= 0 \\ \alpha \quad\;\; + 2\gamma &= 1 \\ 2\alpha - \beta \quad\;\; &= 1 \end{aligned}\right] \qquad \alpha = 0 \quad \beta = 1 \quad \gamma = \tfrac{1}{2}$$

$$\frac{Q}{\rho c_p} = ABC^2 \qquad A^2 = \alpha g B$$

and $\quad AC^2 = \chi$

then $\quad ABC^2 = \chi B = \dfrac{Q}{\rho c_p}$

$$B = \frac{Q}{\rho c_p} \frac{1}{\chi}$$

$$A = \left(\frac{Q}{\rho c_p}\right)^{1/2} (\alpha g)^{1/2} \chi^{-1/2}$$

$$C = \left(\frac{Q}{\rho c_p}\right)^{-1/4} (\alpha g)^{-1/4} \chi^{3/4}$$

then

$$v_z = \left(\frac{Q}{\rho c_p}\right)^{1/2} (\alpha g)^{1/2}\, \chi^{-1/2} \qquad \text{\underline{indep. of}}\ z$$

$$\Delta T = \frac{Q}{\rho c_p}\,\frac{1}{\chi}\,\frac{1}{z}$$

$$\delta = \left(\frac{Q}{\rho c_p}\right)^{-1/4} (\alpha g)^{-1/4}\, \chi^{3/4}\, z^{1/2}$$