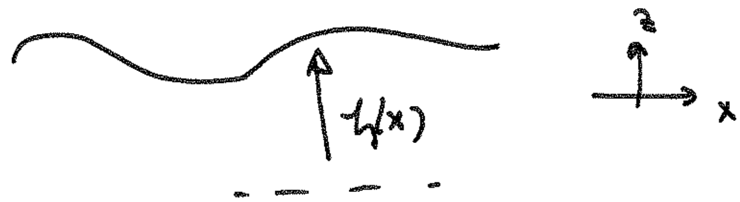
Physics 211 – Problem Set #5
Solutions

1.) a.) For an interface in 2-dimensions



the length of the interface is
$$L = \int [(dx)^2 + (dz)^2]^{1/2}$$
$$= \int dx \left[1 + \left(\frac{dh}{dx}\right)^2\right]^{1/2}$$

For an interface $z = h(x)$ in 3-dimensions, the area of the interface is this value times the extent in y.

Then the interfacial tension is
$$\Delta E = \gamma \int dx\, dy \left[1 + \left(\frac{dh}{dx}\right)^2\right]^{1/2}$$
$$\cong \gamma \int dx\, dy \left[1 + \frac{1}{2}\left(\frac{dh}{dx}\right)^2 + \cdots\right]$$

For a general 2-d surface in 3 dimensions, a similar formula holds. The derivation is harder; we must divide the surface into parallelograms and compute their areas.

There is an "automatic" method to do this. Write the path length on the interface as

$$ds^2 = g_{ij}\, dx^i dx^j$$

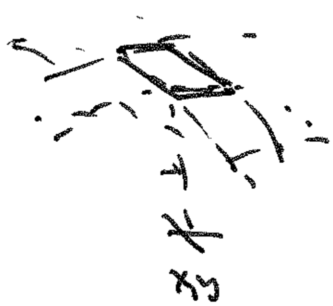
$$= \left| \hat{x}\, dx + \hat{y}\, dy + \hat{z}\left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy\right)\right|^2$$

$$= \left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right] dx^2 + 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\, dx\, dy + \left[1 + \left(\frac{\partial h}{\partial y}\right)^2\right] dy^2$$

so that

$$g_{ij} = \begin{pmatrix} 1 + \left(\frac{\partial h}{\partial x}\right)^2 & \frac{\partial h}{\partial x}\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x}\frac{\partial h}{\partial y} & 1 + \left(\frac{\partial h}{\partial y}\right)^2 \end{pmatrix}$$

If we diagonalize g, it has eigenvalues $\lambda_1^2\, \lambda_2^2$ at x, y. A small parallelogram at $x\,y$ has area $\lambda_1 \lambda_2\, dx\, dy$



This is $d(\text{area}) = \sqrt{\det g}\; dx\, dy$. We can compute $\det g$ and expand it to leading order in h.

$$\sqrt{\det g} = \left[\left(1 + \left(\frac{\partial h}{\partial x}\right)^2\right)\left(1 + \left(\frac{\partial h}{\partial y}\right)^2\right) - \left(\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\right)^2\right]^{\frac{1}{2}}$$

$$\cong 1 + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial h}{\partial y}\right)^2 + \cdots$$

More formally if $g = 1 + \Delta$

$$[\det g]^{1/2} = [1 + \mathrm{tr}\,\Delta + \dots]^{1/2} = 1 + \tfrac{1}{2}\mathrm{tr}\,\Delta + \dots$$

Either way

$$\Delta E = \gamma \int dx\,dy \left[1 + \tfrac{1}{2}(\vec{\nabla}_\perp h)^2 + \dots \right]$$

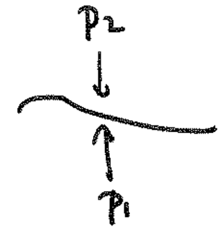
A small variation of the surface changes the energy by

$$\delta E = \gamma \int dx\,dy \; \vec{\nabla} h \cdot \vec{\nabla} \delta h$$

$$= \int dx\,dy \, (-\gamma \nabla^2 h) \, \delta h$$

A small variation of the surface changes the energy by this effect and by doing work against the pressure difference

$$\delta E = \int dx\,dy \; \delta h \left[(p_2 - p_1) - \gamma \nabla_\perp^2 h \right]$$



so the equilibrium is

$$p_2 - p_1 - \gamma \nabla^2 h = 0$$

A surface bent like:

has $\frac{d^2 h}{dx^2} < 0$ and so is in equilibrium with a $p_1 > p_2$.

b.) In our analysis of the Kelvin-Helmholtz instability, we had three boundary conditions:

$$\frac{\partial}{\partial z}\delta\phi_2 = \frac{\partial \eta}{\partial t} + V_2 \frac{\partial \eta}{\partial x}$$

$$\frac{\partial}{\partial z}\delta\phi_1 = \frac{\partial \eta}{\partial t} + V_1 \frac{\partial \eta}{\partial x}$$

$$\rho_1 \left[V_1 \frac{\partial}{\partial x}\delta\phi_1 + \frac{\partial}{\partial t}\delta\phi_1 + g\eta \right] = \rho_2 \left[V_2 \frac{\partial \delta\phi_2}{\partial x} + \frac{\partial \delta\phi_2}{\partial t} + g\eta \right]$$

the last condition was $-p_1 = -p_2$, so the surface tension modifies it to

$$\rho_1 \left[\left(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x}\right)\delta\phi_1 + g\eta \right] = \rho_2 \left[\left(\frac{\partial}{\partial t} + V_2 \frac{\partial}{\partial x}\right)\delta\phi_2 + g\eta \right] + \gamma \nabla^2 \eta$$

let

$$\delta\phi_1 = \phi_1 e^{-i\omega t + ikx + kz}$$

$$\delta\phi_2 = \phi_2 e^{-i\omega t + ikx - kz}$$

$$\eta = \eta\, e^{-i\omega t + ikx}$$

as before:

$$\phi_2 = +i\left(\frac{\omega - V_2 k}{k}\right)\eta \qquad \phi_1 = -i\left(\frac{\omega - V_1 k}{k}\right)\eta$$

so

$$\left[\rho_2 \frac{(\omega - V_2 k)^2}{k} + \rho_1 \frac{(\omega - V_1 k)^2}{k} - \gamma k^2 + (\rho_2 - \rho_1) g\right]\eta = 0$$

The equation can be solved for $\omega(k)$.

For $\rho_1 = \rho_2 = \rho \qquad V_1 = V_2 = 0$

$$2\rho \frac{\omega^2}{k} - \gamma k^2 = 0$$

$$\omega^2 = \frac{\gamma k^3}{2\rho}$$

e.) For $\rho_1 > \rho_2$, $V_1 = 0$

$$(\rho_1 + \rho_2)\,\omega^2 - 2\rho_2 V_2 \omega k + \rho_2 V_2^2 k^2 = \gamma k^3 + (\rho_1 - \rho_2) k g$$

$$\left(\omega - \frac{\rho_2 V_2 k}{\rho_1 + \rho_2}\right)^2 = \frac{1}{\rho_1 + \rho_2}\left[\gamma k^3 + (\rho_1 - \rho_2) k g - \rho_2 V_2^2 k^2 + \frac{\rho_2^2 (V_2 k)^2}{\rho_1 + \rho_2}\right]$$

$$\left(\omega - \frac{\rho_2 V_2 k}{\rho_1+\rho_2}\right)^2 = \frac{1}{\rho_1+\rho_2}\left[\gamma k^3 + (\rho_1-\rho_2)kg - \frac{\rho_1\rho_2}{\rho_1+\rho_2}V_2^2 k^2\right]$$

$$\omega = \frac{\rho_2 V_2 k}{\rho_1+\rho_2} \pm \frac{1}{(\rho_1+\rho_2)^{1/2}}\left[(\rho_1-\rho_2)kg - \frac{\rho_1\rho_2 (V_2 k)^2}{\rho_1+\rho_2} + \gamma k^3\right]^{1/2}$$

When the quantity in the bracket becomes negative we set

$$\omega = \frac{\rho_2 V_2 k}{\rho_1+\rho_2} \pm i\,\Omega$$

one of which is an unstable mode. For $\gamma = 0$, the bracket is always negative for sufficiently large k. For $\gamma > 0$, $V_2 = 0$, the bracket is always positive. In general, the criterion for stability is

$$\frac{\rho_1\rho_2}{\rho_1+\rho_2}(V_2 k)^2 < \gamma k^3 + (\rho_1-\rho_2)kg$$

$$V_2^2 < \left(\frac{\rho_1+\rho_2}{\rho_1\rho_2}\right)\left[\gamma k + \frac{(\rho_1-\rho_2)g}{k}\right]$$

The right-hand side has a minimum when

$$\gamma - \frac{(\rho_1-\rho_2)g}{k^2} = 0$$

$$k = \left[\frac{(\rho_1-\rho_2)g}{\gamma}\right]^{1/2}$$

so we have stability for all k if

$$V_2^{\,2} < \frac{\rho_1+\rho_2}{\rho_1\rho_2} \cdot 2 \cdot \left[\gamma g (\rho_1-\rho_2)\right]^{1/2}$$

For $\rho_1 \gg \rho_2$

$$V_2 < \left[\frac{2}{\rho_2}\right]^{1/2} \left[\gamma g \rho_1\right]^{1/4}$$

d.) For wind-generated waves on the sea, the instability sets in at

$$V_2 = \left[\frac{2}{1.25\ \mathrm{kg/m^3}}\right]^{1/2} \left[(0.074\ \mathrm{kg/sec^2})(9.8\ \mathrm{m/sec^2})(1000\ \mathrm{kg/m^3})\right]^{1/4}$$

(units are!
$$\frac{\left[\mathrm{kg^2/m^2\,sec^4}\right]^{1/4}}{\left[\mathrm{kg/m^3}\right]^{1/2}} = \mathrm{m/sec} \ !$$

$$V_2 = 6.6\ \mathrm{m/sec}$$

$$\lambda_{crit.} = \frac{2\pi}{k_{crit}} = 2\pi \left[\frac{\gamma}{\rho_1 g}\right]^{1/2} = 2\ \mathrm{mm}$$

stronger winds are needed to generate long-wavelength surface waves.

2.) a.) The volume of the cylinder is $\pi a^2 l$.

Its area is $2\pi a l$. The change in energy if the radius is changed $a \to a + da$ is

$$dE = -(p - p_0)\, 2\pi a l\, da + \gamma\, 2\pi l\, da$$

The equilibrium condition is $dE = 0$

or

$$(p - p_0) = \frac{\gamma}{a}$$

b.) From problem 1, the area of the cylinder is given, to quadratic order in small quantities, by

$$\text{Area} = \int d\phi\, dz \left[\left[\eta^2 + \left(\frac{\partial \eta}{\partial \phi}\right)^2\right]\left[1 + \left(\frac{\partial \eta}{\partial z}\right)^2\right] \cdots\right]^{1/2}$$

$$= \int d\phi\, dz \left(\eta + \frac{1}{2}\frac{1}{\eta}\left(\frac{\partial \eta}{\partial \phi}\right)^2 + \frac{1}{2}\eta\left(\frac{\partial \eta}{\partial z}\right)^2 + \cdots\right)$$

$$\eta = a + \delta\eta$$

$$= \int d\phi\, dz\, a \left[1 + \frac{\delta\eta}{a} + \frac{1}{2}\frac{1}{a^2}\left(\frac{\partial \delta\eta}{\partial \phi}\right)^2 + \frac{1}{2}\left(\frac{\partial \delta\eta}{\partial z}\right)^2 + \cdots\right]$$

Now change $\delta\eta$ by $d\eta$. The change in energy is

$$dE = \int d\phi\, dz \left\{ -(p-p_0)(a+\delta h)\, dh - \delta p\, a\, dh + \gamma \left[dh - dh \frac{1}{a} \frac{\partial^2}{\partial \phi^2} \delta h - a\, dh \frac{\partial^2 \delta h}{\partial z^2} \right] \right\}$$

or, for equilibrium:

$$\delta p = -\frac{(p-p_0)(a+\delta h)}{a} + \frac{\gamma}{a}\left[1 - \frac{1}{a}\frac{\partial^2 \delta h}{\partial \phi^2} - a \frac{\partial^2 \delta h}{\partial z^2} \right]$$

$$= -\frac{\gamma}{a}\left(1 + \frac{\delta h}{a}\right) + \frac{\gamma}{a} - \gamma \frac{1}{a^2} \frac{\partial^2 \delta h}{\partial \phi^2} - \gamma \frac{\partial^2 \delta h}{\partial z^2}$$

$$\delta p = -\gamma \left[\frac{\delta h}{a^2} + \frac{\partial^2}{\partial z^2} \delta h + \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} \delta h \right]$$

The boundary condition on the velocity is that the radial velocity at the surface should match the motion of the surface.

So

$$v_r = \frac{\partial}{\partial t} \delta h$$

For $\eta = 0$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p \qquad \vec{\nabla} \cdot \vec{v} = 0$$

linearize:

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho} \vec{\nabla} \delta p \qquad \vec{\nabla} \cdot \vec{v} = 0$$

c.) Assume potential flow $\vec{v} = \vec{\nabla}\psi$

Then $\vec{\nabla}\cdot\vec{v} = 0 \Rightarrow \nabla^2\psi = 0$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\vec{\nabla}\delta p \Rightarrow \frac{\partial}{\partial t}\vec{\nabla}\psi = -\frac{1}{\rho}\vec{\nabla}\delta p$$

$$\nabla^2\delta p = -\rho\frac{\partial}{\partial t}\nabla^2\psi = 0$$

Let $\delta p = p(r)\, e^{-i\omega t + ikz + im\phi}$ etc

$$\nabla^2\delta p = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} p(r) - k^2 p(r) - \frac{m^2}{r^2}p(r) = 0$$

again

$$\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \left(k^2 + \frac{m^2}{r^2}\right)\right] p(r) = 0$$

d.) This is the equation of the <u>modified</u> <u>Bessel functions</u> $I_n(kr)$, $K_n(kr)$. $p(r)$ must be regular at $r=0$,

hence $p(r) = A\, I_n(kr)$

Now reconstruct $\vec{v}$

$$\frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}\frac{p}{\rho} \quad \text{so} \quad -i\omega\,\vec{v}(r) = -\frac{1}{\rho}\vec{\nabla}p(r)$$

For v_r

$$v_r = -i \frac{1}{\omega\rho} \frac{\partial}{\partial r} p(r)$$

$$= -i \frac{k}{\omega\rho} A\, I_n'(kr)$$

also

$$v_r = \frac{\partial}{\partial t} \delta h \qquad \text{at } r = a$$

$$= -i\omega h$$

so $$h = \frac{k}{\omega^2\rho} A\, I_n'(ka)$$

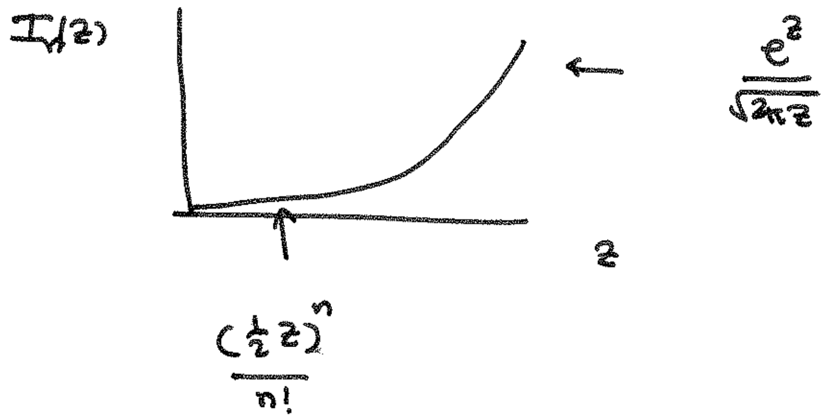
We can now write the boundary condition on the pressure on p.9 as

$$A\, I_n(ka) = -\gamma \frac{k}{\omega^2\rho} A\, I_n'(ka) \left[\frac{1}{a^2} - k^2 - \frac{n^2}{a^2} \right]$$

so that

$$\omega^2 = \left[(ka)^2 + (n^2 - 1) \right] \frac{\gamma k}{a^2 \rho} \frac{I_n'(ka)}{I_n(ka)}$$

e.) You can look up the behavior of $I_n(z)$ in your favorite handbook. You will find:



$I_n(z) > 0$ $\quad I_n'(z) > 0$ for z real and positive, so $I_n'/I_n > 0$. Then ω^2 is positive if

$$(ka)^2 + n^2 - 1 > 0$$

there is no instability for $n = \pm 1, \pm 2, \ldots$

But, for $n = 0$, there is an instability if

$$(ka)^2 - 1 < 0 \quad \text{or} \quad k < \frac{1}{a}$$

3.) This problem is an application of the stability analysis for inviscid Couette flow done in class. We derived, for axisymmetric perturbations,

$$(-DD_* + k^2)\, v_r = \frac{k^2}{\omega^2}\, \Phi\, v_r$$

where $\Phi = \frac{1}{r^3}\frac{d}{dr}(rV)^2 \qquad D = \frac{d}{dr} \qquad D_* = \frac{d}{dr} + \frac{1}{r}$

& $v_r = 0$ on the boundaries.

For a bulk rotation $\quad V = \Omega r \quad , \quad \Phi = 4\Omega^2$

Then the above equation becomes:

$$\left[-\frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right) + k^2 - k^2 \frac{\Omega^2}{\omega^2} \right] v_r = 0$$

or

$$\left[-\frac{d}{dr}\frac{1}{r}\frac{d}{dr} r + k^2\left(1 - \frac{\Omega^2}{\omega^2}\right)\right] v_r = 0$$

write $\quad v_r = \frac{d}{dr} f \quad$; then f satisfies

$$\left[-\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + k^2\left(1 - \frac{\Omega^2}{\omega^2}\right)\right] f = 0$$

This is Bessel's equation. The eigenfunctions are:

$J_0(\varkappa r)$, $\varkappa^2 = k^2\left(\frac{\Omega^2}{\omega^2} - 1\right)$

since $\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + \varkappa^2\right] J_0(\varkappa r) = 0$

then
$$\frac{\Omega^2}{\omega^2} = k^2 + \varkappa^2$$

$$\omega^2 = \frac{k^2}{k^2+\varkappa^2}\,\Omega^2$$

$\varkappa$ is determined by the condition

$v_r = 0$ at $r = a$ $\Rightarrow$ $J_0'(\varkappa a) = 0$

[I have also imposed that v_r is regular at $r=0$; that is why J_0 and not Y_0 is used.]

Now $J_0'(z) = -J_1(z)$

so if z_m is the m^{th} zero of $J_1(z)$

$$\omega_m = \frac{k}{\left[k^2 + z_m^2/a^2\right]^{1/2}}\,\Omega$$