Physics 211 – Problem Set # 4

Solutions

1.) a.) For Poiseuille flow,

$$V(r) = V_0 (1 - r^2/R^2)$$

In term of V_0, the mass flow is

$$Q/\rho = \int d(\text{area})\ V_0 (1 - r^2/R^2)$$

$$= \int_0^R dr\ 2\pi r\ V_0\ (1 - r^2/R^2) \qquad x = r^2/R^2$$

$$= \pi V_0 R^2 \int_0^1 dx\ (1-x)$$

$$= \tfrac{1}{2}\pi R^2 V_0$$

At the front of the pipe, we have a velocity uniform over the pipe; then

$$Q/\rho = \pi R^2 V_{in}$$

so $$V_{in} = \tfrac{1}{2} V_0$$

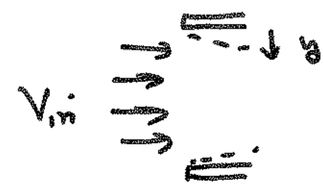
b.) the force on the pipe is

$$dF = d(\text{area})(-2\eta\, \sigma^{zr}) = d(\text{area})\left(-\eta \frac{\partial v^z}{\partial r}\Big|_{r=R}\right)$$

far down the pipe this is

$$dF = 2\pi R\, dl \cdot \eta \cdot 2 V_0 \frac{1}{R}$$

Near the front of the pipe, we can use the formulae for a boundary layer derived in class. If the boundary layer is thin, we can ignore the curvature of the boundary.



$$y = R - r$$

$$v_z = \frac{\partial \psi}{\partial y}$$

$$\psi = \text{const.} \sqrt{2z}\, f(w)$$

where $\quad w = \dfrac{y}{\sqrt{2z}}\sqrt{\dfrac{V_{in}}{\nu}} \quad$ so $\quad w \sim \dfrac{y}{\sqrt{R}} \leftarrow$ Reynolds no.

$f(w)$ is the Blasius function $\quad f'(w) \to 1$ as $w \to \infty$

$$v_z = (\text{const}) \sqrt{\frac{V_{in}}{\nu}}\, f'(w) \longrightarrow V_{in} \text{ as } y \to \infty$$

so $\quad (\text{const}) = \sqrt{\nu V_{in}}$

$$v_z = V_{in}\, f'(w)$$

$$-\frac{\partial}{\partial r} v_z \Big|_{y=0} = \frac{V_{in}^{3/2}}{\sqrt{2z\nu}} f''(w)\Big|_{w=0} = \frac{V_{in}^{3/2}}{\sqrt{2z\nu}} f''(0)$$

For the Blasius function $f''(0) = \alpha = 0.4696..$

$$dF = 2\pi R \; dz \; \eta \; \frac{V_{in}^{3/2}}{\sqrt{2z\nu}} \cdot \alpha$$

c.) This formula holds out to $z \sim z_*$ at which the boundary layer thickness is $\sim$ Radius

$$R \sim \sqrt{\frac{2z_*\nu}{V_{in}}}$$

$$z_* \sim \frac{V_{in}}{2\nu} R^2$$

Integrate the result of part (b): from the boundary layer region

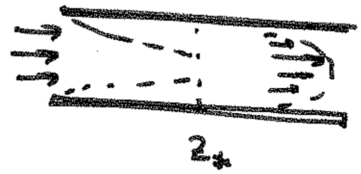
$$F = 2\pi R \int_0^{z_*} dz \; \eta \; \frac{\alpha V_{in}^{3/2}}{\sqrt{2\nu}} \cdot \frac{1}{\sqrt{z}}$$

$$= 2\pi R \; \alpha\eta \; \frac{V_{in}^{3/2}}{\sqrt{2\nu}} \; 2\sqrt{z_*}$$

$$\cong 2\pi R \alpha \eta \frac{V_{in}^{3/2}}{\sqrt{2\nu}} \sqrt{\frac{V_{in}}{2\nu}} \; R$$

$$= \pi R^2 \cdot \eta \cdot \alpha \; \frac{V_{in}^2}{\nu}$$

d.) The total force on the pipe has two components



Force from region of Poiseuille flow:

$$F = 4\pi\eta\, V_0\,(l - z_*) = 4\pi\eta \frac{2Q}{\rho}\frac{1}{\pi R^2}(l - z_*)$$

$$= 8\,\frac{Q\nu}{R^2}(l - z_*)$$

Force from region of boundary layer

$$F = 2\pi R \cdot \alpha\eta \cdot \frac{V_{in}^{3/2}}{\sqrt{2\nu}}\sqrt{z_*}$$

$$= \frac{4\alpha\eta}{R}\,\frac{Q}{\rho}\,\frac{V_{in}^{1/2}}{\sqrt{2\nu}}\sqrt{z_*} = \frac{4\alpha}{R}\,Q\nu\left(\frac{V_{in} z_*}{2\nu}\right)^{1/2}$$

so

$$F = 8\,\frac{Q\nu}{R^2}\left[\,l - z_* + \frac{\alpha}{2}\,R\,\underbrace{\left(\frac{V_{in} z_*}{2\nu}\right)^{1/2}}_{(\text{Reynold no.})^{1/2}}\,\right]$$

For small z_*, $z_* \ll R$, the third term is larger than the second by $(\text{Reynolds no})^{1/2} / \left(\frac{z_*}{R}\right)^{1/2}$

However, I estimated z_* by

$$\frac{z_*}{R} \sim (\text{Reynolds no.})$$

so then two terms are comparable in size.

The presence of the boundary layer region must increase the force on the pipe for a given Q, since $\frac{\partial v_z}{\partial r}$ is larger in the region near the front of the pipe. So we must have

$$(P-P_0)\pi R^2 = F = 8\,\frac{Q\nu}{R^2}\,(l+\Delta l) \qquad l_{eff} = l + \Delta l$$

$$\text{where} \qquad \Delta l > 0 \quad \text{and} \qquad |\Delta l| \sim z_* \sim \frac{V_{in} R}{2\nu}\cdot R$$

2.) The differential equation is

$$x^2 \frac{dy}{dx} = 0$$

$$+ \epsilon \left[xy^2 - x + y^3 - x^2 \frac{dy}{dx} \right]$$

$$+ \epsilon^2 \left[-xy^2 - x + 2y^2 \right]$$

a.) Write $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$

y_0 satisfies $x^2 \dfrac{dy_0}{dx} = 0$ or $\dfrac{dy_0}{dx} = 0$

with $y_0(1) = 1$

so $y_0(x) = 1$

y_1 satisfies $x^2 \dfrac{dy_1}{dx} = \left[x y_0^2 - x + y_0^3 - x^2 \dfrac{dy_0}{dx}\right]$

$= 1$

and $y_1(1) = 0$

$x^2 \dfrac{dy_1}{dx} = 1 \qquad \dfrac{dy_1}{dx} = \dfrac{1}{x^2} \qquad y_1 = C - \dfrac{1}{x}$

$y_1(x) = 1 - \dfrac{1}{x}$

y_2 ~~satisfies~~ $y_2(1) = 0$ and

$$x^2 \frac{dy_2}{dx} = -xy_0^2 - x + 2y_0^2 + 2xy_0y_1 + 3y_0^2 y_1 - x^2 \frac{dy_1}{dx}$$

$$x^2 \frac{dy_2}{dx} = -2x + 2 + 2x\left(1 - \frac{1}{x}\right) + 3\left(1 - \frac{1}{x}\right) - 1$$

$$= 2 - \frac{3}{x}$$

$$\frac{dy_2}{dx} = \frac{2}{x^2} - \frac{3}{x^3}$$

$$y_2 = C - \frac{2}{x} + \frac{3}{2}\frac{1}{x^2}$$

$$y_2 = \frac{1}{2} - \frac{2}{x} + \frac{3}{2}\frac{1}{x^2}$$

$\therefore$ all

$$y = 1 - \epsilon\left[\frac{1}{x} - 1\right] + \epsilon^2\left[\frac{3}{2x^2} - \frac{2}{x} + \frac{1}{2}\right]$$

$$+ O(\epsilon^3)$$

b.) the successive terms behave as $\left(\frac{\epsilon}{x}\right)^n$, so the perturbation solution breaks down when $x \sim \epsilon$.

c.) As $x \to 0$ the leading terms are

$$x^2 \frac{dy}{dx} = \epsilon y^3$$

Let $w = x/\epsilon^\alpha$

$$\epsilon^\alpha w^2 \frac{dy}{dw} = \epsilon y^3$$

these balance when $\alpha = 1$

so let $\omega = x/\epsilon$ $\quad x = \epsilon\omega$ and
rewrite the differential equation

$$(1+\epsilon)\,\epsilon\,\omega^2 \frac{dy}{d\omega} = \epsilon\left[(1-\epsilon)\epsilon\omega y^2 - \epsilon(1+\epsilon)\omega + y^3 + 2\epsilon y^2\right]$$

now organize in powers of ϵ

$$\left(\omega^2 \frac{dy}{d\omega} - y^3\right) = \epsilon\left[\omega y^2 - \omega + 2y^2 - \omega^2\frac{dy}{d\omega}\right] + \epsilon^2\left[-\omega y^2 - \omega\right]$$

write $\quad y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$

d.) Y_0 satisfies

$$\omega^2 \frac{dY_0}{d\omega} = Y_0^3$$

$$\frac{dY_0}{Y_0^3} = \frac{d\omega}{\omega^2}$$

$$-\frac{1}{2Y_0^2} = -\frac{1}{\omega} - C$$

$$2Y_0^2 = \frac{\omega}{1+\omega C}$$

$$Y_0 = \left(\frac{w}{2 + 2wC}\right)^{1/2}$$

For $\omega \gg 1$ $\quad Y_0 \to \left(\frac{1}{2C}\right)^{1/2}$ so setting $2C = 1$

matches the large x solution $y_0 = 1$

See what happens if we expand further:

$$Y_0 = \left(\frac{\omega}{2+\omega}\right)^{1/2} = \frac{1}{(1 + 2/\omega)^{1/2}}$$

$$\omega \to \infty \qquad = 1 - \frac{1}{2}\frac{2}{\omega} + \frac{3}{8}\left(\frac{2}{\omega}\right)^2 + \cdots$$

$$= 1 - \frac{1}{\omega} + \frac{3}{2}\frac{1}{\omega^2} + \cdots$$

$$Y_0 = 1 - \frac{\epsilon}{x} + \frac{3}{2}\frac{\epsilon^2}{x^2} + \cdots$$

this accounts for all of the most singular terms in the result on p. 8

e.) Y_1 satisfies

$$\omega^2 \frac{dY_1}{d\omega} - 3 Y_0^2 Y_1 = \omega Y_0^2 - \omega + 2 Y_0^2 - \omega^2 \frac{dY_0}{d\omega}$$

$$\omega^2 \frac{dY_1}{d\omega} - \frac{3\omega}{2+\omega} Y_1 = (2+\omega)\left(\frac{\omega}{2+\omega}\right) - \omega - \left(\frac{\omega}{2+\omega}\right)^{3/2}$$

$$= - \cdot \left(\frac{\omega}{2+\omega}\right)^{3/2}$$

It is useful to substitute $v = \frac{1}{\omega}$ $\frac{\omega}{2+\omega} = \frac{1}{1+2v}$

$$-\frac{dY_1}{dv} - \frac{3}{1+2v} Y_1 = -\left(\frac{1}{1+2v}\right)^{3/2}$$

$$\frac{1}{(1+2v)^{3/2}} \frac{d}{dv}\left((1+2v)^{3/2} Y_1\right) = \left(\frac{1}{1+2v}\right)^{3/2}$$

$$\frac{d}{dv}\left((1+2v)^{3/2} Y_1\right) = 1$$

$$Y_1 = \frac{v+A}{(1+2v)^{3/2}}$$

expand for large ω, $v \to 0$ $v = \frac{\epsilon}{x}$

$$Y_1 = A + v - \frac{3}{2} A \cdot 2v + O(v^2)$$

as $v \to 0$ $\epsilon Y_1 \to \epsilon A$

this should match the term $(-\epsilon)(-1)$ in the expansion on p. 7, so $A = 1$

then

$$\epsilon Y_1 = \epsilon(1 - 2v) + \cdots$$

$$= +\epsilon - \epsilon^2 \frac{2}{x} + \cdots$$

this picks up the second most singular terms on p. 7

Now restore in Y_0, Y_1 : $\quad \omega = x/\epsilon \qquad \nu = \epsilon/x$

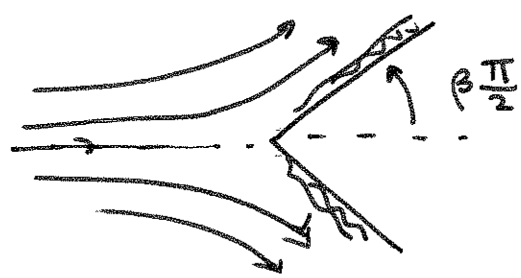
$$Y_0 = \left(\frac{x}{x+2\epsilon}\right)^{1/2} \qquad Y_1 = \frac{1+\epsilon/x}{(1+2\epsilon/x)^{3/2}} = \frac{x^{1/2}(x+\epsilon)}{(x+2\epsilon)^{3/2}}$$

then

$$y = \left(\frac{x}{x+2\epsilon}\right)^{1/2} + \epsilon\,\frac{x^{1/2}(x+\epsilon)}{(x+2\epsilon)^{3/2}} + \mathcal{O}(\epsilon^2)$$

where now the error term is <u>uniformly</u> $\mathcal{O}(\epsilon^2)$ for $0<x<1$.

3.)



a.) Work in cylindrical coordinates.
Represent the velocity field by a stream funct. ψ

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\phi} \qquad v_\phi = -\frac{\partial}{\partial r}\psi$$

so that $\quad \vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r} r v_r + \frac{1}{r}\frac{\partial}{\partial\phi} v_\phi = 0$

and $\quad v_\phi = 0 \quad$ on $\quad \phi = \pm\frac{\beta\pi}{2}$

For potential flow ψ is harmonic

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]\psi = 0$$

$v_\phi = 0$ at $\phi = \pm\frac{\beta\pi}{2}$ so try

$$\psi = f(r) \quad \sin\left(\frac{\phi - \beta\pi/2}{1-\beta/2}\right)$$

then also $\psi = 0$ $v_\phi = 0$ at $\phi = \pi$ as required.

$f(r)$ satisfies

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{(1-\beta/2)^2}\frac{1}{r^2}\right] f(r) = 0$$

$$f = r^\alpha \quad \Rightarrow \quad \alpha^2 = \frac{1}{(1-\beta/2)}$$

then

$$\psi = a\, r^{1/(1-\beta/2)} \quad \sin\left(\frac{\phi - \frac{\beta\pi}{2}}{1-\beta/2}\right)$$

along the wedge,

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\phi} = \frac{a}{1-\beta/2}\, r^{\frac{\beta/2}{1-\beta/2}}$$

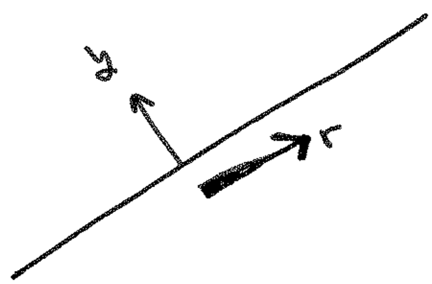
Write this as

$$v_r = v_0\, r^\gamma \quad \text{on the wedge}$$

$$\gamma = \frac{\beta/2}{1-\beta/2}$$

then $\psi = (1-\beta/2)\, v_0\, r^{\gamma+1} \sin\left(\frac{\phi - \beta\pi/2}{1-\beta/2}\right)$

b.) Now try to make a theory of the boundary layer.
r is the coordinate along the wedge. Let y
be an orthogonal coordinate $\quad y = r\phi$



For $\nu \to 0$ the boundary layer should be very thin.
Outside this layer, v_r should go to $v_0 r^\gamma$ and
ψ should go to the form above.

Start from the Navier-Stokes equation in the form

$$\vec{v}\cdot\vec{\nabla}\,\omega = \nu \nabla^2 \omega$$

where ω is vorticity $\quad \omega = -\nabla^2\psi$

For a thin boundary layer $\quad \nabla^2 \approx \frac{d^2}{dy^2}$

$$\vec{v}\cdot\vec{\nabla} = v_r \frac{\partial}{\partial r} + v_y \frac{\partial}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial}{\partial r} + \left(-\frac{\partial\psi}{\partial r}\right)\frac{\partial}{\partial y}$$

so the vorticity equation becomes

$$\psi_y \psi_{ryy} - \psi_r \psi_{yyy} = \nu \psi_{yyyy}$$

this is

$$\frac{\partial}{\partial y}\left(\psi_y \psi_{ry} - \psi_r \psi_{yy} - \nu \psi_{yyy}\right) = 0$$

or

$$\psi_y \psi_{ry} - \psi_r \psi_{yy} - \nu \psi_{yyy} = (\text{indep of } y)$$

As for a flat plate, we want to balance r and y derivatives by expanding the y dimension by a factor $1/\sqrt{\nu}$

so, look for a solution of the form

$$\psi = C\, r^{\eta} f(z) \qquad z = \frac{y}{A\sqrt{\nu}\, r^{\theta}}$$

$$v_r = \frac{\partial \psi}{\partial y} = \frac{C}{A\sqrt{\nu}}\, r^{\eta - \theta} f'(z)$$

To set $v_r = v_\phi = 0$ at $y = 0$ we need $f(0) = f'(0) = 0$

To match the boundary condition as $y/\sqrt{\nu} \to \infty$, we need

$$\frac{C}{A\sqrt{\nu}} = v_0 \qquad r^{\eta - \theta} = r^{\gamma}$$

$$\text{or} \qquad \eta - \theta = \frac{\beta/2}{1 - \beta/2} = \frac{\beta}{2 - \beta}$$

Now
$$\psi_y = \frac{c}{A\sqrt{\nu}}\, r^{\zeta-\theta} f'$$

$$\psi_r = c\zeta r^{\zeta-1} f - c\theta \frac{y}{A\sqrt{\nu}} r^{\zeta-\theta-1} f'$$

$$= c\, r^{\zeta-1}\left[\zeta f - \theta z f'\right]$$

$$\psi_{ry} = \frac{c}{A\sqrt{\nu}}\, r^{\zeta-\theta-1}\left[(\zeta-\theta) f' - \theta z f''\right]$$

then

$$\psi_y \psi_{ry} - \psi_r \psi_{yy} - \nu \psi_{yyy}$$

$$= \frac{c^2}{(A\sqrt{\nu})^2}\left(r^{\zeta-\theta} f'\right) r^{\zeta-\theta-1}\left((\zeta-\theta) f' - \theta z f''\right)$$

$$- \frac{c^2 r^{\zeta-1}}{(A\sqrt{\nu})^2}\left(\zeta f - \theta z f'\right) r^{\zeta-2\theta} f''$$

$$- \nu \frac{c}{(A\sqrt{\nu})^3}\, r^{\zeta-3\theta} f'''$$

$$= \frac{c^2}{(A\sqrt{\nu})^2}\, r^{2\zeta-2\theta-1}\left((\zeta-\theta)(f')^2 - \zeta f f''\right)$$

$$- \nu \frac{c}{(A\sqrt{\nu})^3}\, r^{\zeta-3\theta} f'''$$

to balance these terms, we need $\quad 2\zeta - 2\theta - 1 = \zeta - 3\theta$

or $\quad \zeta + \theta = 1 \quad$ together with $\quad \zeta - \theta = \frac{\beta}{2-\beta}$

so $\quad \zeta = \frac{1}{2-\beta} \qquad \theta = \frac{1-\beta}{2-\beta} \qquad \zeta - \theta = \frac{\beta}{2-\beta}$

then

$$\psi_y \psi_{ry} - \psi_r \psi_{yy} - \nu \psi_{yyy}$$

$$= -\frac{\nu C}{(A\sqrt{\nu})^3} r^{\zeta - 3\theta} \left[f''' + \frac{CA\sqrt{\nu}}{\nu}\left[\zeta f f'' - (\zeta - \theta)(f')^2\right]\right]$$

so set

$$\frac{CA\sqrt{\nu}}{\nu} \frac{1}{2-\beta} = 1 \quad \text{together with} \quad \frac{C}{A\sqrt{\nu}} = U_0$$

$$C = \left[(2-\beta)\nu U_0\right]^{1/2} \qquad A = \left[\frac{2-\beta}{U_0}\right]^{1/2}$$

then

$$f''' + f f'' - \beta (f')^2 = \text{indep. of } y$$

as $\quad y \to \infty \quad f' \to 1 \quad f'' \to 0$

so the right-hand side equals $(-\beta)$

The solution is $\longrightarrow$

$$v_r = v_0 r^\gamma f'(z) \qquad z = y\left[\frac{v_0}{(2-\beta)\nu}\right]^{\frac{1}{2}} \frac{1}{r^{(1-\beta)/2-\beta}}$$

where

$$f''' + f f'' + \beta\left[1-(f')^2\right] = 0$$

with $\quad f(0) = f'(0) = 0 \qquad f'(z\to\infty) = 1$

c) The force per unit area on the wedge is

$$F/\text{area} = \eta \left.\frac{\partial v_r}{\partial y}\right|_{y=0}$$

$$= \eta\, v_0 \left[\frac{v_0}{2-\beta\nu}\right]^{\frac{1}{2}} r^{\gamma-2\theta} f''(0)$$

$$\gamma - 2\theta = \frac{\beta-(1-\beta)}{2-\beta} = \frac{2\beta-1}{2-\beta}$$

so $\quad \dfrac{dF}{dr} \sim r^{(2\beta-1)/2-\beta}$

for $\beta\to 0$ the exponent $\to -\frac{1}{2}$ as required.

more generally $\quad \gamma - 2\theta = -\frac{1}{2} + \frac{3}{2}\,\frac{\beta}{2-\beta} > -\frac{1}{2}$,

a softer dependence.