Physics 211 - Problem Set #3
Solutions

1.) The Navier-Stokes equation with steady flow is

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{V}$$

Look for a solution of the form $\vec{V} = v(r)\hat{\phi}$ $p = p(r)$

$$\vec{V}\cdot\vec{\nabla} = v(r)\frac{1}{r}\frac{\partial}{\partial\phi}$$

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = v(r)\frac{1}{r}v(r)\frac{\partial}{\partial\phi}\hat{\phi} = -\frac{v^2(r)}{r}\hat{r}$$

$$-\frac{1}{\rho}\vec{\nabla}p = -\hat{r}\frac{1}{\rho}\frac{\partial}{\partial r}p$$

$$\nabla^2\vec{V} = \left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]v(r)\hat{\phi} \qquad \frac{\partial^2}{\partial\phi^2}\hat{\phi} = -\hat{\phi}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\right]v(r)\cdot\hat{\phi}$$

so

$$-\frac{v^2}{r}\hat{r} = -\frac{1}{\rho}\frac{\partial}{\partial r}p\,\hat{r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\right]v(r)\cdot\hat{\phi}$$

to obtain a solution we must satisfy

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{pressure balances centrifugal force}$$

and

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2}\right] v(r) = 0$$

The solution of the second equation is

$$v(r) = ar + \frac{b}{r}$$

Given $v(r)$, we can use the first equation to compute the pressure

$$\frac{\partial}{\partial r} p = \rho\left[a^2 r + \frac{2ab}{r} + \frac{b^2}{r^3}\right]$$

$$p = p_0 + \rho\left(\frac{a^2}{2} r^2 + 2ab \lg r - \frac{b^2}{2r^2}\right)$$

b.) $v(R_1) = \Omega_1 R_1 \quad v(R_2) = \Omega_2 R_2$

then $$aR_1 + \frac{b}{R_1} = \Omega_1 R_1 \qquad aR_2 + \frac{b}{R_2} = \Omega_2 R_2$$

$$a(R_1^2 - R_2^2) = \Omega_1 R_1^2 - \Omega_2 R_2^2$$

$$b\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) = \Omega_1 - \Omega_2$$

so

$$a = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \qquad b = \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

$$v(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\, r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}\, \frac{1}{r}$$

$$\frac{p(r)}{\rho} = (\text{const}) + \frac{1}{(R_2^2 - R_1^2)^2} \Big\{ \tfrac{1}{2}(\Omega_2 R_2^2 - \Omega_1 R_1^2)^2 r$$

$$+ 2(\Omega_2 R_2^2 - \Omega_1 R_1^2)(\Omega_1 - \Omega_2) R_1^2 R_2^2 \ln r$$

$$- \tfrac{1}{2}(\Omega_1 - \Omega_2)^2 R_1^4 R_2^4 \frac{1}{r^2} \Big\}$$

c.) The force per unit area on the cylinders comes from both stress forces and pressure. However, the pressure force is $\parallel \hat{r}$ and exerts no torque.

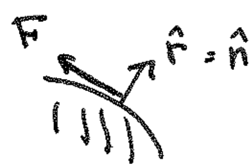
The stress comes from

$$\nabla^i v^j = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}\right)^i (v(r)\hat{\phi})^j$$

$$= \hat{r}^i \frac{\partial v}{\partial r} \hat{\phi}^j - \hat{\phi}^i \frac{v(r)}{r} \hat{r}^j$$

$$2\sigma^{ij} = (\hat{r}^i \hat{\phi}^j + \hat{r}^j \hat{\phi}^i)\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right]$$

The tangential force per unit area is



$$\frac{\hat{\phi}\cdot F}{\text{Area}} = 2\eta\, \hat{r}^i \hat{\phi}^j \sigma^{ij} = \eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

so the

inner cylinder $\quad \dfrac{\vec{F}\cdot\hat{\phi}}{\text{Area}} = \eta\left(\dfrac{\partial v}{\partial r} - \dfrac{v}{r}\right)\Big|_{r=R_1}$

outer cylinder $\quad \dfrac{\vec{F}\cdot\hat{\phi}}{\text{Area}} = -\eta\left(\dfrac{\partial v}{\partial r} - \dfrac{v}{r}\right)\Big|_{r=R_2}$

Note that if $v = \Omega r$, the fluid is in simple bulk rotation and exerts no torque.

For $v(r)$ on p. 3

$$\frac{\partial v}{\partial r} - \frac{v}{r} = \frac{2(\Omega_2 - \Omega_1) R_1^2 R_2^2}{R_2^2 - R_1^2} \frac{1}{r^2}$$

The torques are $\qquad l =$ height of cylinder

inner cylinder: $\quad \tau = (2\pi R_1 l)\cdot R_1 \cdot \eta \dfrac{2(\Omega_2-\Omega_1)R_1^2R_2^2}{(R_2^2 - R_1^2)} \dfrac{1}{R_1^2}$

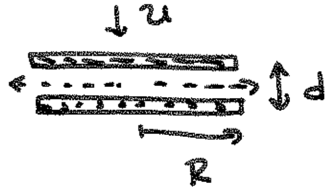
outer cylinder: $\quad \tau = -(2\pi R_2 l) R_2 \, \eta \dfrac{2(\Omega_2-\Omega_1)R_1^2R_2^2}{R_2^2 - R_1^2} \dfrac{1}{R_2^2}$

so

$$\tau_{\text{inner}} = -\tau_{\text{outer}} = 4\pi l \eta \frac{(\Omega_2 - \Omega_1) R_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

$$> 0 \text{ if } \Omega_2 - \Omega_1 > 0 \quad \checkmark$$

2.)



a) For a symmetric flow pattern

$$\vec{v} = v_r \hat{r} + v_z \hat{z} \qquad p \text{ is indep of } \phi$$

$$0 = \vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z} v_z$$

Then we can estimate

$$v_z \sim u \qquad \frac{\partial v_z}{\partial z} \sim \frac{u}{d} \qquad v_r \sim \frac{R}{d} u$$

b.) For a steady out-flow, the Navier Stokes equation is

$$0 = -\frac{1}{\rho}\vec{\nabla} p + \nu \nabla^2 \vec{v}$$

$$\nabla^2 \vec{v} = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right)(v_r \hat{r} + v_z \hat{z})$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) v_r \hat{r}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) v_z \hat{z}$$

The largest terms are the terms with $\frac{\partial^2}{\partial z^2}$.

so z compont : $\frac{1}{\rho}\frac{\partial}{\partial z}p \cong \nu \frac{\partial^2}{\partial z^2} V_z$

r compnt : $\frac{1}{\rho}\frac{\partial}{\partial r}p \cong \nu \frac{\partial^2}{\partial z^2} V_r$

so $$|\partial p/\partial z| \,/\, \left|\frac{\partial p}{\partial r}\right| \sim \frac{V_z}{V_r} \sim \frac{d}{R} \ll 1.$$

c.) The r compnt can be solved by assuming that p(r) changes slowly on the scale of d. Then

$$\frac{\partial^2}{\partial z^2} V_r = \frac{1}{\rho\nu}\frac{\partial p}{\partial r}$$

with $V_r = 0$ at $z=0$, $z=d$

$$V_r = \frac{1}{2\rho\nu}\left(z^2 + \text{const}\cdot z\right)\cdot\frac{\partial p}{\partial r}$$

$$V_r = \frac{1}{2\eta}\, z\,(d-z)\cdot\left(-\frac{\partial p}{\partial r}\right)$$

d.) from part (a)

$$V_z = -\int_0^z dz\, \frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \text{const.}$$

If $V_z = 0$ at $z=0$, the const. is zero

If $V_z = -u$ at $z=d$ $\quad u = \int_0^d dz\, \frac{1}{r}\frac{\partial}{\partial r}(r V_z)$

then

$$u = \int_0^d dz \ \frac{1}{2\eta} z(d-z) \cdot \left[-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r} \right]$$

$$= \frac{1}{2\eta} \cdot \frac{d^3}{6} \cdot \left(-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r} \right)$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r} = -\frac{12\eta u}{d^3}$$

$$p(r) = -\frac{3\eta u}{d^3} r^2 + \text{const.}$$

If $p(r)$ is high in the center and negligible at the edge of the disk,

$$p(r) = \frac{3\eta u}{d^3}(R^2 - r^2)$$

e.) The force on the top disk receives contributions from pressure and from shear forces. But

$$p \sim \frac{\eta u}{d^3} R^2 \qquad \eta \sigma_{zz} \sim \eta \frac{u}{d}$$

so pressure is much more important. The force due to pressure is:

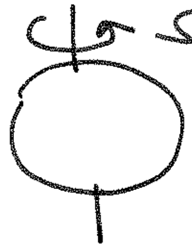
$$F = \int d^2a \; p$$

$$= \int_0^R dr \; 2\pi r \cdot \frac{3\eta u}{d^3} (R^2 - r^2)$$

$$= \frac{3\eta u}{d^3} \pi \int_0^{R^2} dr^2 \; R^2 - r^2$$

$$F = \frac{3\pi}{2} \frac{\eta u}{d^3}$$

3.) Choose $\vec{\Omega} \parallel \hat{z}$



a.) The solution for $\vec{v}$ should be of the form $\vec{v} = v_\phi \hat{\phi}$

with $\hat{\phi} = (-\sin\phi, \cos\phi, 0)$

$$\nabla^2 \vec{v} = \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{d}{d\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) \vec{v}$$

$$= \left[(\nabla^2 v_\phi) - \frac{1}{r^2 \sin^2\theta} v_\phi \right] \hat{\phi}$$

The Navier Stokes equation is

$$0 = -\frac{1}{\rho} \vec{\nabla} p + \nu \nabla^2 \vec{v}$$

or $\quad \vec{\nabla} p = \rho\nu \nabla^2 \vec{v} = $ vector in $\hat{\phi}$ direction

But p is independent of ϕ by symmetry, so $p = (\text{const})$

when then need to solve $\nabla^2 \vec{V} = 0$

b.) If $\vec{v} = \vec{\nabla} \times (f(r)\vec{\Omega}) = (\hat{r} \times \vec{\Omega}) \frac{\partial f}{\partial r}$

$$\hat{r} \times \Omega = - \sin\theta \cdot \hat{\phi}$$

so this points in the $\hat{\phi}$ direction. Also, $\vec{\nabla} \cdot (\vec{\nabla} \times f\vec{\Omega}) = 0$
so $\vec{\nabla} \cdot \vec{v} = 0$

$$\nabla^2 \vec{\nabla} \times (f(r)\vec{\Omega}) = \vec{\nabla} \times (\nabla^2 f(r) \cdot \vec{\Omega}) = 0$$

is $\nabla^2 f = 0$. So in this case the above expression solves the Navier - Stokes equation.

c.) For $f(r) = \frac{c}{r}$

$$\vec{v} = \frac{c}{r^2} \vec{\Omega} \times \hat{r} = \frac{c}{r^2} \sin\theta \hat{\phi}$$

For $C = R^3$

$$\vec{v} = R \sin\theta \hat{\phi} = \vec{\Omega} \times \vec{r}$$

which is just the speed with which a point on the surface of the sphere is rotating. Thus, the fluid velocity with respect to the sphere is zero, which is the boundary condition.

d.) The friction force on the sphere is

$$dF^i = d(\text{Area}) \cdot 2\eta\, \sigma^{ij} \hat{r}^j \Big|_{r=R}$$

Compute σ^{ij}

$$\nabla^i v^j = \nabla^i \frac{R^3}{r^3} (\Omega \times r)^j$$

$$= \nabla^i \frac{R^3}{r^3} \epsilon^{jkl} \Omega^k r^l$$

$$= -3 \frac{r^i}{r^5} R^3 \epsilon^{jkl} \Omega^k r^l + \delta^{il} \epsilon^{jkl} \frac{R^3}{r^3} \Omega^k$$

$$= + \epsilon^{ijk} \frac{R^3}{r^3} \Omega^k - 3 \frac{R^3}{r^5} r^i \epsilon^{jkl} \Omega^k r^l$$

$$2\sigma^{ij} = \nabla^i v^j + \nabla^j v^i$$

$$= -3 \frac{R^3}{r^5} (r^i \epsilon^{jkl} + r^j \epsilon^{ikl}) \Omega^k r^l$$

the first term in $\nabla^i v^j$ is antisymmetric and cancels out.

$$2\sigma^{ij} \hat{r}^j = -3 \frac{R^3}{r^5} \cdot \frac{r^2}{r} \epsilon^{ikl} \Omega^k r^l$$

$$= -3 \frac{R^3}{r^4} (\vec{\Omega} \times \vec{r})^i$$

so

$$d\vec{F} = d(\text{Area}) \cdot -3\eta \left(\frac{\vec{\Omega} \times \vec{r}}{r} \right)$$

the torque on the sphere is

$$\vec{\tau} = \int \vec{r} \times d\vec{F} = \int d(\text{area})\ (-3\eta)\ \vec{r} \times (\vec{\Omega} \times \hat{r})\Big|_{r=R}$$

$$= \int d(\text{area})\ (-3\eta)\ [\vec{\Omega}\, r - \hat{r}\,(\vec{\Omega}\cdot\vec{r})]\Big|_{r=R}$$

$$= \int d(\text{area})\ (-3\eta)[\vec{\Omega} R - \hat{r}\,\hat{r}\cdot\vec{\Omega}\cdot R]$$

$$= 4\pi R^2\ (-3\eta)(\vec{\Omega}\, R)\cdot\left(1 - \tfrac{1}{3}\right)$$

$$\vec{\tau} = -8\pi R^3 \eta\, \vec{\Omega}$$

e.) Now the equation of motion is

$$\frac{\partial \vec{v}}{\partial t} = \nu \nabla^2 \vec{v}$$

Look for a solution of the form

$$\vec{v} = \mathrm{Re}\left[\ \vec{\nabla} \times f(r)\cdot \vec{\Omega}_0\, e^{-i\omega t}\ \right]$$

Again, this solves $\vec{\nabla}\cdot\vec{v} = 0$

$$\frac{\partial \vec{v}}{\partial t} - \nu\nabla^2\vec{v} = \mathrm{Re}\left[\vec{\nabla} \times (-i\omega - \nu\nabla^2) f\ \ \vec{\Omega}_0\right]$$

so this solves the equation if

$$\left(\nabla^2 + i\frac{\omega}{\nu}\right) f = 0$$

f.) A simple solution of $(\nabla^2 + k^2) f = 0$ is

$$f = a \frac{e^{ikr}}{r}$$

Here $\quad k = \left(i\frac{\omega}{\nu}\right)^{1/2} = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{\omega}{\nu}\right)^{1/2}$

The + square root gives $f(r) \to 0$ as $r \to \infty$

[Proof of solution: $\quad \nabla^2 \frac{e^{ikr}}{r} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-e^{ikr} + ikr\, e^{ikr}\right)$$

$$= -\frac{ik}{r^2} e^{ikr} + \frac{ik}{r^2} e^{ikr} - k^2 \frac{e^{ikr}}{r}$$

$$= -k^2 \frac{e^{ikr}}{r} \qquad \checkmark \quad]$$

then

$$\vec{v} = \mathrm{Re}\left[\vec{\nabla} f \times \vec{\Omega}_0\, e^{-i\omega t}\right]$$

$$= \mathrm{Re}\left[(\vec{\Omega}_0 \times \hat{r})\, e^{-i\omega t}\, a\left(\frac{1}{r^2} - i\frac{k}{r}\right) e^{ikr}\right]$$

$$= \mathrm{Re}\left[(\vec{\Omega}_0 \times \vec{r})\, e^{-i\omega t}\, a\left(\frac{1-ikr}{r^3}\right) e^{ikr}\right]$$

the boundary condition is

$$\vec{v} = \mathrm{Re}\ \vec{\Omega}_0 \times \vec{r}\, e^{-i\omega t} \quad \text{at } r = R$$

so

$$a = \frac{R^3}{1-ikR} e^{-ikR}$$

g.) As $\omega \to 0$, $k \to 0$ and we find $a = R^3$
as in part (c.)

h.) $k^2 = i\frac{\omega}{\nu}$ so $kR \to \infty$ as $\omega \to \infty$

Then

$$\nabla^i v^j = \mathrm{Re}\ \nabla^i\left[(\vec{\Omega}_0\times\vec{r})^j \frac{R^3}{r^3}\frac{1-ikr}{1-ikR}e^{ik(r-R)}e^{-i\omega t}\right]$$

$$= \mathrm{Re}\ (\vec{\Omega}_0\times\vec{r})^j \frac{R^3}{r^3}\ \hat{r}^i\cdot(ik)\ \frac{1-ikr}{1-ikR}e^{ik(r-R)}e^{-i\omega t}$$

$$+\ (\text{term of order } 1 \text{ as } k\to\infty)$$

$$\nabla^i v^j\Big|_{r=R} = \mathrm{Re}\ (\vec{\Omega}_0\times\vec{r})^j\ \hat{r}^i\ (ik)\ e^{-i\omega t}$$

$$2\sigma^{ij}\hat{r}^j = \mathrm{Re}\ (\vec{\Omega}_0\times\vec{r})^i\ (ik)\ e^{-i\omega t}$$

then
$$\vec{\tau} = \int \vec{r}\times d\vec{F}$$

$$= \int d(\text{area})\,\mathrm{Re}\left[(ik)e^{-i\omega t}\right]\cdot\eta\cdot\ \vec{r}\times(\vec{\Omega}_0\times\vec{r})$$

$$= \int d(\text{area})\ \eta\ \mathrm{Re}\left[ik\,e^{-i\omega t}\right](\vec{\Omega}_0 r^2 - \vec{r}\,\vec{r}\cdot\vec{\Omega})$$

$$= 4\pi R^2\ \eta\ \mathrm{Re}\left[ik\,e^{-i\omega t}\right]\vec{\Omega}_0 R^2\cdot\frac{2}{3}$$

Now $\quad \mathrm{Re}\; ik\, e^{-i\omega t}$

$$= \mathrm{Re}\;\; i\; e^{i\pi/4} \left(\frac{\omega}{\nu}\right)^{1/2} e^{-i\omega t}$$

$$= \mathrm{Re} \left(\frac{\omega}{\nu}\right)^{1/2} e^{-i\left(\omega t - \frac{3\pi}{4}\right)}$$

So for $\omega \to \infty$

$$\vec{\tau} \sim \frac{8\pi}{3}\, \eta R^4 \left(\frac{\omega}{\nu}\right)^{1/2} \cos\left(\omega t - \frac{3\pi}{4}\right) \vec{\Omega}_0$$