Physics 211 — Problem Set #2

Solutions

1.) Starting from the Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi$$

$$\vec{\omega} = \vec{\nabla}\times\vec{v} \quad \Rightarrow \quad \vec{v}\times\vec{\omega} = \frac{1}{2}\vec{\nabla}v^2 - (\vec{v}\cdot\vec{\nabla})\vec{v}$$

In steady flow $\frac{\partial \vec{v}}{\partial t} = 0$. Then

$$\vec{v}\cdot\vec{\nabla}\vec{v} = \frac{1}{2}\vec{\nabla}v^2 - \vec{v}\times\vec{\omega} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi$$

also

$$\vec{\nabla}h = T\vec{\nabla}s + \frac{1}{\rho}\vec{\nabla}p$$

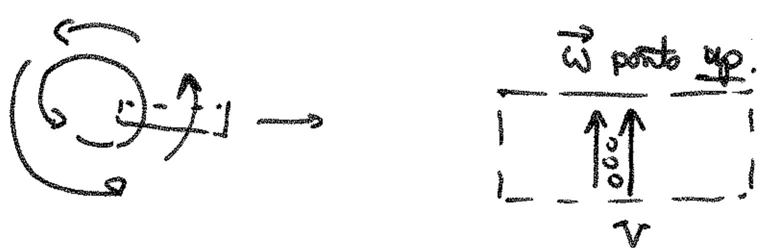
then

$$\vec{\nabla}\left(\frac{1}{2}v^2 + h + \Phi\right) = T\vec{\nabla}s + \vec{v}\times\vec{\omega}$$

that is

$$\vec{\nabla}B = T\vec{\nabla}s + \vec{v}\times\vec{\omega}$$

Now look at a tornado from above:



so $\vec{v}\times\vec{\omega}$ points outward

If we ignore changes in s, B increases as we go outward.

Now $B \approx \frac{1}{2}v^2 + \frac{p}{\rho}$, so if B is small in the center, p is small.

An estimate of $\int d\vec{r} \cdot \vec{v} \times \vec{\omega}$ is $\sim v^2$

For $v \sim 100$ m/sec $\qquad v^2 \sim 10^8$ cm^2/sec^2

Air at room temperature has $\rho \sim 1$ g/ℓ $\sim 10^{-3}$ g/cm^3

$$\frac{\Delta p}{\rho} \sim v^2 \quad \text{gives} \quad \Delta p \sim 10^5 \text{ g cm}^2/\text{sec}/\text{cm}^3$$

$$\sim 0.1 \text{ atm}$$

So a rapidly spinning tornado can give a drop in air pressure comparable to 1 atm.

2.) a.) If water is approximately incompressible, the same amount of mass must pass through each spherical shell as the bubble collapses. If the collapse is spherically symmetric, this amount is

$$Q = \rho\, 4\pi r^2\, v(r)$$

so $$v(r) = \frac{1}{r^2} \cdot F(t)$$

then the Euler equation (ignoring Φ) gives

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}.\vec{\nabla}\,\vec{v} = -\frac{1}{\rho}\vec{\nabla}p$$

the radial component is $\qquad \vec{v}.\vec{\nabla} = v\frac{\partial}{\partial r}$

$$\frac{\partial v}{\partial t} + v\frac{\partial}{\partial r}v = -\frac{1}{\rho}\frac{\partial}{\partial r}p$$

or $$\frac{1}{r^2}\frac{\partial F}{\partial t} + v\frac{\partial}{\partial r}v + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$+\frac{\partial}{\partial r}\left(-\frac{\partial F(t)/\partial t}{r} + \frac{1}{2}v^2 + \frac{p}{\rho}\right) = 0$$

b.) Integrate this from R to ∞, with $p(R)=0$, $v(\infty)=0$

$$-\frac{1}{R}\frac{\partial F}{\partial t} + \frac{1}{2}v^2(R) + 0 = \frac{p_0}{\rho}$$

c.) Now $v(R) = \dfrac{F(t)}{R^2}$, but $v(R)$ is also $\dfrac{dR}{dt}$

then $$\frac{d}{dt} = v(R)\frac{d}{dR}$$

$$\frac{1}{R}\frac{dF}{dt} = \frac{v}{R}\frac{dF}{dR} = \frac{F}{R^3}\frac{dF}{dR} = \frac{1}{2R^3}\frac{d}{dR}F^2$$

so we obtain $$-\frac{1}{R^3}\frac{d}{dR}F^2 + \frac{F^2}{R^4} = \frac{2p_0}{\rho}$$

We can integrate this equation

$$\frac{1}{R^2}\left[-\frac{d}{dR}\left(\frac{F^2}{R}\right)\right] = \frac{2p_0}{\rho}$$

$$\frac{d}{dR}\left(\frac{F^2}{R}\right) = -\frac{2p_0}{\rho}R^2$$

$$\frac{F^2}{R} = C - \frac{2}{3}\frac{p_0}{\rho}R^3$$

at $t=0$ $R=R_0$ $V=0$ $F=0$. Then

$$\frac{F^2}{R} = \frac{2p_0}{3\rho}\left(R_0^3 - R^3\right)$$

$$F = \left[\frac{2p_0}{3\rho}\left(\frac{R_0^3}{R^2} - 1\right)R^4\right]^{\frac{1}{2}}$$

then

$$V(R) = \left(\frac{2p_0}{3\rho}\right)^{\frac{1}{2}}\left(\frac{R_0^3}{R^3}-1\right)^{\frac{1}{2}}$$

3.) a.) $$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$w - 1 = \frac{1}{2}\left(z - 2 + \frac{1}{z}\right) = \frac{(z-1)^2}{2z}$$

$$w + 1 = \frac{1}{2}\left(z + 2 + \frac{1}{z}\right) = \frac{(z+1)^2}{2z}$$

then

$$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2$$

b.) For $z \sim 1$, $w \sim 1$

$$(w-1) \sim \frac{1}{2}(z-1)^2$$

this transforms. $z = 1 + \epsilon e^{i\alpha}$ to $w = 1 + \frac{1}{2}\epsilon^2 e^{2i\alpha}$

so angles are doubled at $z = 1$

c.) Let $z = \rho e^{i\phi}$ for fixed ρ and $\phi \in [0, 2\pi]$

This is a circle in the z plane.

$$w = \frac{1}{2}\left(\rho e^{i\phi} + \frac{1}{\rho} e^{-i\phi}\right)$$

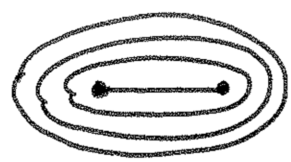
$$= \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\phi + \frac{i}{2}\left(\rho - \frac{1}{\rho}\right)\sin\phi$$

Write this as $\omega = x+iy \qquad x = \frac{\rho+1/\rho}{2}\cos\phi \qquad y = \frac{\rho-1/\rho}{2}\sin\phi$

If $\quad a = \frac{1}{2}(\rho+\frac{1}{\rho}) \quad b = \frac{1}{2}(\rho-\frac{1}{\rho})$ then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These are ellipses with foci at $\pm[a^2-b^2]^{\frac{1}{2}} = \pm 1$



Circles in the z-plane with radius ρ and $1/\rho$ give the same ellipse. The degenerate case is $\rho = 1$; this gives $a=1$, $b=0$.

d.) Now let $z = \rho e^{i\phi}$ with ϕ fixed and $\rho \in [0,\infty]$

Again $\quad \omega = \frac{1}{2}(\rho+\frac{1}{\rho})\cos\phi + \frac{i}{2}(\rho-\frac{1}{\rho})\sin\phi$

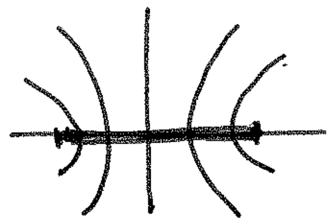
Let $\rho = e^{\eta} \qquad \eta \in [-\infty,\infty]$

$$\omega = \cosh\eta\cos\phi + i\sinh\eta\sin\phi$$

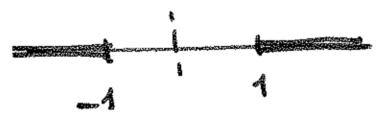
If we write $\omega = x+iy \qquad x = \cosh\eta\cos\phi \qquad y = \sinh\eta\sin\phi$

then $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a = \cos\phi \quad b = \sin\phi$

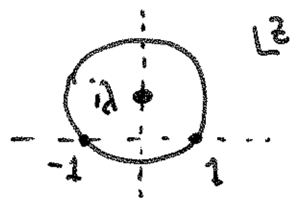
$a^2+b^2 = 1$, so again the foci are at ± 1



ϕ and $-\phi$ give the same hyperbola. $\phi = 0, \pi$ give the degenerate cases



e.) Now consider



Note that

$$
\begin{array}{lccc}
 & \underline{z} & \longrightarrow & \underline{w} \\
 & 1 & \longrightarrow & 1 \\
 & -1 & \longrightarrow & -1 \\
\text{top of circle} & i(\lambda + (\lambda^2+1)^{1/2}) & \longrightarrow & +i\lambda \\
\text{bottom of circle} & i(\lambda - (\lambda^2+1)^{1/2}) & \longrightarrow & +i\lambda
\end{array}
\Bigg] \text{ the same point}
$$

since if $\quad z = i\left[\lambda \pm (\lambda^2+1)^{1/2}\right]$

$$w = \frac{1}{2}\left(i\left[\lambda \pm (\lambda^2+1)^{1/2}\right] + (-i)\,\frac{1}{(\lambda \pm (\lambda^2+1)^{1/2})}\right)$$

$$= \frac{i}{2}\left(\lambda \pm (\lambda^2+1)^{1/2} - \frac{\lambda \mp (\lambda^2+1)^{1/2}}{[\lambda^2 - (\lambda^2+1)]}\right)$$

$$= \frac{i}{2}\left(\lambda \pm (\lambda^2+1)^{1/2} + \lambda \mp (\lambda^2+1)^{1/2}\right) = i\lambda$$

In fact, all points on the circle above the real axis are related to points below the real axis. For

$$z = i\eta e^{i\phi}$$

if z is on the circle

$$|z - i\lambda|^2 = \lambda^2+1 \rightarrow (\eta\cos\phi - \lambda)^2 + (\eta\sin\phi)^2 = \lambda^2+1$$

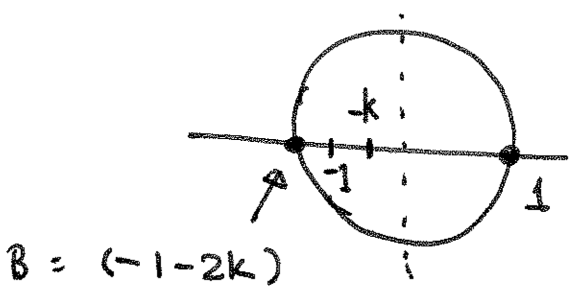
$$\eta^2 - 2\lambda\eta\cos\phi + \cancel{\lambda^2} = \cancel{\lambda^2}+1$$

or

$$\eta - \frac{1}{\eta} = 2\lambda\cos\phi$$

so if $z = i\eta e^{i\phi}$ is on the circle $\frac{1}{z} = -i\frac{1}{\eta}e^{-i\phi}$ is also on the circle. These two points map to the same point in ω.

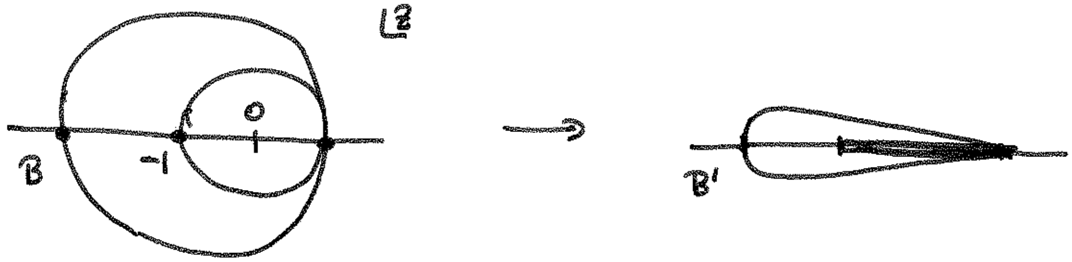
f.) Next, consider the circle with radius $[k+1]$ and center at $z = -k$.



The point $(-1-2k)$ is mapped to a point in ω to the left of $\omega = -1$

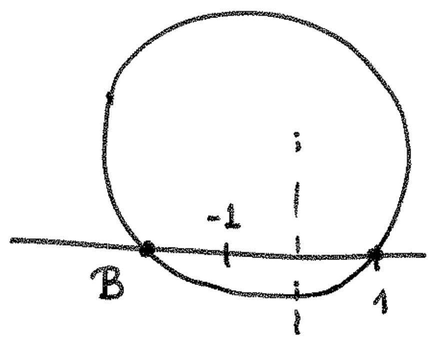
We saw in (b) that angles are doubled at $z=1$

Then

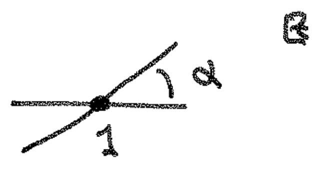


The circle centered at 0 collapses to $[-1,1]$; this is the degenerate limit

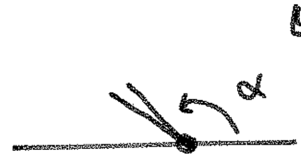
g.) Finally, consider a circle through $z=1$ offset in both ways



Near $z=1$, the circle intersects the real axis at an angle



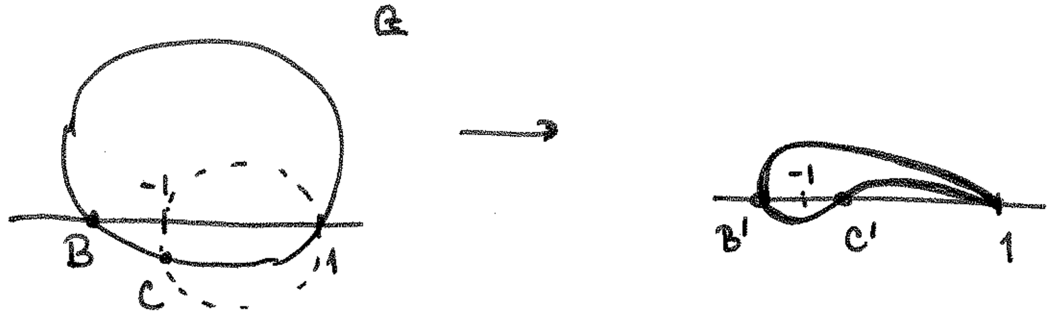
since angles are doubled, this maps to



(2 lines tangent at $w=1$)

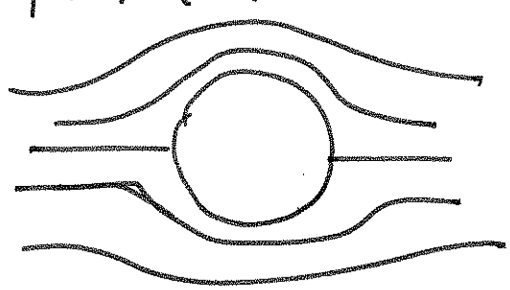
The point B' maps to a point on the real axis to the left of -1. Points outside the circle $|z|=1$ are mapped to the same side of the real axis. Points inside this circle are mapped to the opposite side of the real axis.

Then

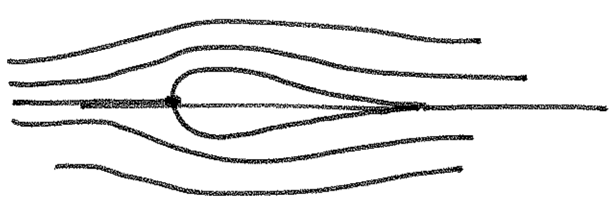


Yes, this looks like an air-foil.

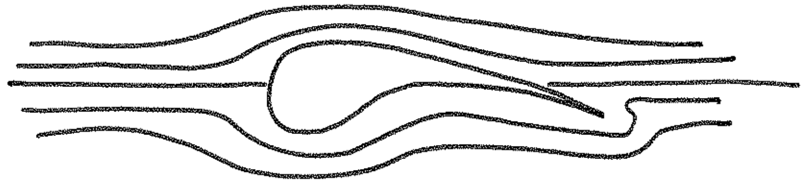
h.) The potential flow around a circle has streamlines



Under a conformal mapping, streamlines map into streamlines, so we find



for the figure in (f)



for the figure in (g)

The first figure has a stagnation point at the front.
At the cusp,

in z $\qquad \psi \sim (\mathrm{Re}\, z - 1)^2 - (\mathrm{Im}\, z)^2$

$$\psi + i\chi \sim (z-1)^2 \sim \omega - 1$$

so in the first figure there is no singularity in the flow at the cusp.

In the second figure, there are stagnation ~~points~~ and the front edge and on the back slope, as indicated.

Near $z=1$ $\qquad \vec{\nabla}\psi \sim (\cos\alpha, \sin\alpha)$

$$\psi + i\chi \sim (z-1)\, e^{-i\alpha} \sim (\omega-1)^{\frac{1}{2}} e^{-i\alpha}$$

For $\omega = 1 + \rho e^{i\phi}$

$$\psi + i\chi \sim \rho^{\frac{1}{2}} e^{-i(\alpha - \phi/2)}$$

$$\psi \sim \rho^{\frac{1}{2}} \cos(\alpha - \phi/2)$$

$$v_r = \frac{\partial}{\partial \rho}\psi \sim \rho^{-\frac{1}{2}} \cos(\alpha - \phi/2)$$

and becomes singular as $\rho \to 0$.

The stagnation point on the top surface and backward flow behind it is also unphysical. Instead we would have "separation".

4.) a.) For a gravity wave, the position of the surface is

$$\eta = A \sin(\omega t - kx)$$

and

$$\psi = A \frac{g}{\omega} \cos(\omega t - kx)\, e^{+kz}$$

with $\omega^2 = gk$ $\qquad \frac{k}{\omega} = \sqrt{\frac{k}{g}}$

The velocity of the fluid is $\vec{v} = \vec{\nabla} \psi$

$$v_x = A g \frac{k}{\omega} \sin(\omega t - kx)\, e^{+kz}$$

$$v_z = A g \frac{k}{\omega} \cos(\omega t - kx)\, e^{kz}$$

These formulae are valid for small amplitude oscillations,

$$A \ll k^{-1}$$

Note $\quad g\frac{k}{\omega} = \sqrt{kg}$

$$\int d^3x \; \tfrac{1}{2}\rho v^2 = \int d^2(\text{Area}) \int_{-\infty}^{0} dz \; \tfrac{1}{2}\rho A^2 k g \, e^{2kz}$$

$$= (\text{Area}) \cdot \tfrac{1}{2}\rho A^2 k g \cdot \frac{1}{2k}$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho g A^2$$

$$\int d^3x \; \rho\Phi = \int dx\,dy \cdot \int_0^{\eta} dz \; \rho g z$$

$$= \int dx\,dy \; \tfrac{1}{2}\rho g \eta^2$$

$$= \int dx\,dy \; \tfrac{1}{2}\rho g A^2 \sin^2(\omega t - kx)$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho g A^2$$

so the kinetic and potential energies in the wave are equal and both equal

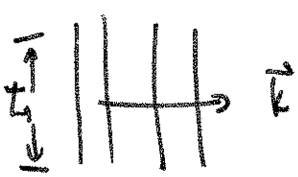
$$\frac{T}{\text{Area}} = \frac{V}{\text{Area}} = \tfrac{1}{4}\rho g A^2$$

one wavelength is $\quad \lambda = \frac{2\pi}{k} = \frac{2\pi g}{\omega^2}$

so

$$E/\text{wavelength} = \pi\rho \frac{g^2}{\omega^2} A^2 \cdot L$$

where L is the orthogonal dimension



b.) Now consider a gravity wave in a fluid of finite depth l. Again

$$\eta = A \sin(\omega t - kx)$$

$$\psi = a \cos(\omega t - kx)\, f(z)$$

and, since $\nabla^2\psi = 0$, again $\frac{d^2}{dz^2} f - k^2 f = 0$

But now we must choose $f(z)$ so that

$$v_z = \frac{\partial}{\partial z}\psi = 0 \quad \text{at} \quad z = -l$$

so $$f(z) = \frac{\cosh k(z+l)}{\cosh kl} \qquad \text{so} \quad f(0) = 1$$

The boundary conditions at $z=0$ are

$$v_z\Big|_{z=0} = \frac{\partial\psi}{\partial z}\Big|_{z=0} = \frac{\partial\eta}{\partial t} \qquad \text{and} \qquad \frac{\partial\psi}{\partial t}\Big|_{z=0} + g\eta = 0$$

then $$ak \frac{\sinh k(z+l)}{\cosh kl}\Big|_{z=0} = \omega A$$

$$-a\omega + gA = 0$$

so $$\omega^2 = gk \tanh kl \qquad a = \frac{gA}{\omega}$$

e.) Now we must recompute T and V.

The computation of V is exactly as before

The computation of T becomes :

$$v_x = A g \frac{k}{\omega} \sin(\omega t - kx) \frac{\cosh k(z+l)}{\cosh kl}$$

$$v_z = A g \frac{k}{\omega} \cos(\omega t - kx) \frac{\sinh k(z+l)}{\cosh kl}$$

$$\int d^3x \ \tfrac{1}{2}\rho v^2 = \int dx\,dy \ \tfrac{1}{2}\rho A^2 \left(\frac{gk}{\omega}\right)^2 \int_{-l}^{0} dz \ \frac{\sin^2(\omega t - kx)\cosh^2 + \cos^2 \sinh^2}{\cosh^2 kl}$$

$$= (\text{Area}) \ \tfrac{1}{4}\rho A^2 \left(\frac{gk}{\omega}\right)^2 \int_{-l}^{0} dz \ \frac{\cosh^2 k(z+l) + \sinh^2 k(z+l)}{\cosh^2 kl}$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho A^2 \left(\frac{gk}{\omega}\right)^2 \frac{1}{k} \int_{0}^{kl} dx \ \frac{\cosh^2 x + \sinh^2 x}{\cosh^2 kl}$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho A^2 \left(\frac{gk}{\omega}\right)^2 \frac{1}{k} \int_{0}^{kl} dx \ \frac{\cosh 2x}{\cosh^2 kl}$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho A^2 \left(\frac{gk}{\omega}\right)^2 \frac{1}{2k} \ \frac{\sinh 2kl}{\cosh^2 kl}$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho A^2 \left(\frac{gk}{\omega}\right)^2 \frac{\sinh kl}{k \cosh kl}$$

and $$\frac{g^2}{\omega^2} k = \frac{g}{\tanh kl}$$

$$= (\text{Area}) \cdot \tfrac{1}{4}\rho A^2 g$$

so again

$$\frac{T}{\text{Area}} = \frac{V}{\text{Area}} = \frac{1}{4} \rho g A^2$$

now

$$\lambda = \frac{2\pi}{k} = \frac{2\pi g}{\omega^2} \tanh kl$$

so

$$E/I_c \;/\text{wavelength} = \pi \rho \frac{g^2}{\omega^2} \tanh(k(\omega) l) \cdot A^2$$

d.) In the shallow-water limit

$$\tanh kl \sim kl$$

$$\omega^2 = gl\, k^2$$

$$E/I_c \;/\text{wavelength} = \pi \rho \frac{(g^3 l)^{1/2}}{\omega} A^2$$

e.) Now let a deep-water wave approach a beach.

If the approach is adiabatic,

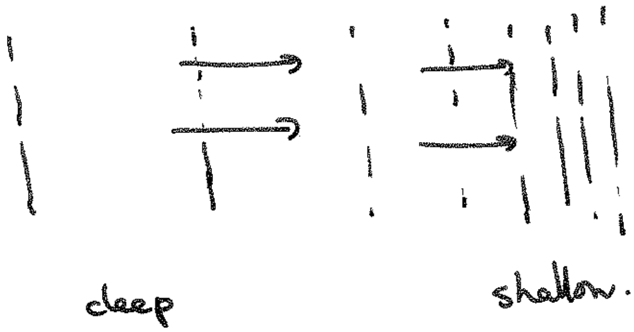
ω is fixed and

E/I_c /wavelength is fixed

deep water : $k = \omega^2/g$

shallow water : $k = \dfrac{\omega}{\sqrt{gl}} = \dfrac{\omega^2}{g} \dfrac{1}{(kl)}$

so k increases, λ decreases

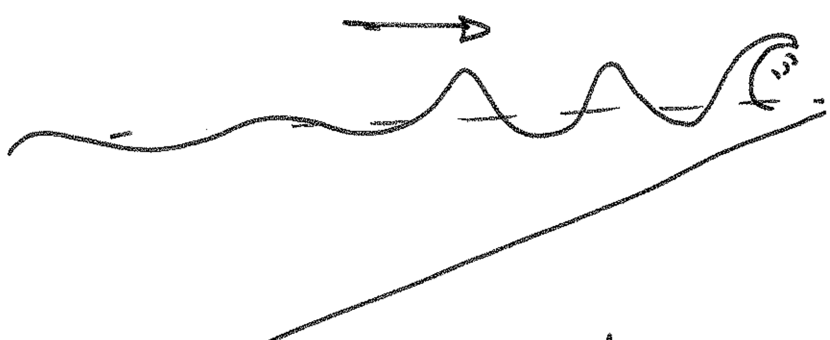


If the energy/wave length remains constant, A must increase:

$$A^2(\ell) = \frac{1}{(k\ell)} A^2_{deep}$$

$$= \left(\frac{g}{\ell}\right)^{1/2} \frac{1}{\omega} A^2_{deep}$$

so $A \sim \frac{1}{\ell^{1/4}}$ as $\ell \to 0$ until the wave motion becomes nonlinear.



Deep water waves with low frequency and large energy/wavelength become tsunamis.