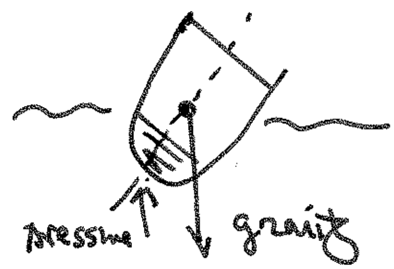
Physics 211 – Problem Set #1
Solutions

1.) When the boat tips over, there are two torques that act on it, one due to gravity, one due to pressure



The gravitational torque is given by

$$\vec{\tau}_g = \int_{\text{boat}} d^3x \;\; \vec{x} \cdot \rho(\vec{x}) \times \vec{g}$$

where $\vec{g} = (0, 0, -g)$ and $\rho(\vec{x})$ is the density of stuff in the boat. This is just

$$\vec{\tau}_g = M \vec{X}_{COG} \times \vec{g}$$

where $\vec{X}_{COG}$ is the center of gravity of the boat.

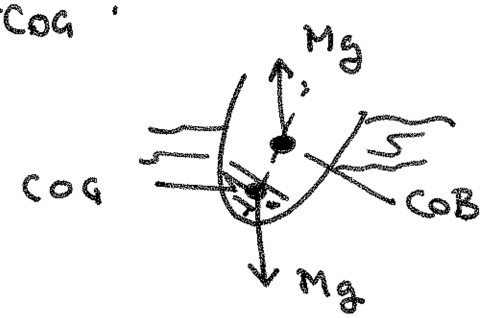
By Archimedes' law, the pressure forces on the boat compensate the force of gravity on the object obtained by replacing the boat by water and air

Ignoring the air, the torque due to the force is then

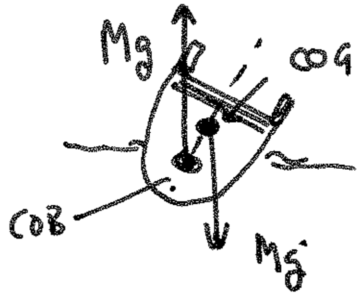
$$\vec{\tau}_p = -\int_{\substack{\text{submerged}\\ \text{part of boat}}} d^3x \quad \vec{x} \;\rho\; \times \vec{g} \qquad \rho = \text{density of water}$$

$$= -M \vec{X}_{COB} \times \vec{g}$$

where $\int_{\text{submerged}} d^3x \,\rho = M$ and the above equation defines the <u>center of buoyancy</u>. The two torques can then be thought to act at $\vec{X}_{COG}$ and $\vec{X}_{COB}$. If $\vec{X}_{COB}$ is <u>higher</u> than $\vec{X}_{COG}$:



we get a <u>restoring</u> torque. If $\vec{X}_{COB}$ is <u>lower</u> than $\vec{X}_{COG}$



the boat is un<u>stable</u>.

2.) a.) Let the mass of the planet be M. The potential in the rotating frame is

$$\Phi = -\frac{G_N M}{r} - \frac{1}{2}\Omega^2 (x^2+y^2)$$

The surface of the planet should be an equipotential

$$\Phi(\text{pole}) = -\frac{G_N M}{R_p}$$

$$\Phi(\text{equator}) = -\frac{G_N M}{R_e} - \frac{1}{2}\Omega^2 R_e^2$$

If the centrifugal force is a small perturbation

$$R_e = R_p + \Delta r$$

$$-\frac{G_N M}{R_p} = -\frac{G_N M}{R_p + \Delta r} - \frac{1}{2}\Omega^2 (R_p + \Delta r)^2$$

$$= -\frac{G_N M}{R_p} + \frac{G_N M}{R_p^2}\Delta r - \frac{1}{2}\Omega^2 R_p^2 + \ldots$$

so

$$\Delta r = \left(\frac{G_N M}{R_p^2}\right)^{-1} \cdot \frac{1}{2}\Omega^2 R_p^2$$

but

$$g = \frac{G_N M}{R_p^2}$$

so

$$\Delta r = R_e - R_p = \frac{\Omega^2 R_p^2}{2g}$$

b.) For a nonrotating planet of uniform density

$$\nabla^2 \Phi = 4\pi G_N \rho$$

The planet should be spherically symmetric : $\Phi = \Phi(r)$

so

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\Phi = 4\pi G_N \rho$$

The solution is $\Phi = A r^2$ with $6A = 4\pi G_N \rho$

so

$$\Phi = \frac{2\pi G_N \rho}{3} r^2 \quad (+\text{ const})$$

c.) $\mu = \sin(\text{latitude})$
$= \cos\Theta$

We expect that a rotating planet will distort:

symmetrically under $\cos\Theta \to -\cos\Theta$.

$P_1(\cos\Theta)$ has the wrong symmetry ; $P_2(\cos\Theta) = +P_2(-\cos\Theta)$

The distorted planet still has uniform density inside, so still $\nabla^2\Phi = 4\pi G\rho$ inside.

Then, for a slowly rotating planet, Φ inside is given by the previous formula plus a solution to $\nabla^2 \Phi = 0$.
I will represent this solution by a multipole expansion

$$\Phi_{in} = \frac{2\pi G_N \rho}{3} r^2 + A + B r^2 P_2(\cos\theta) + O(r^4 P_4(\cos\theta))$$

omitting all terms odd under $\cos\theta \to -\cos\theta$ and all terms singular as $r \to 0$.

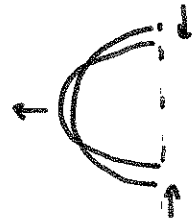
c.) Outside the planet Φ satisfies $\nabla^2 \Phi = 0$. So again we can write a multipole expansion

$$\Phi_{out} = -\left(\frac{C}{r} + \frac{D}{r^3} P_2(\cos\theta) + \cdots\right)$$

omitting all odd terms.

d.) We now need to match the two potentials across the boundary. To do this, we need to know where the boundary is. An ansatz, in the spirit of the multipole expansions, is

$$R(\theta) = R - \delta R \; P_2(\cos\theta)$$



$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$

Notice that the total volume of the planet does not change:

$$V = \int_{-1}^{1} d\cos\Theta \;\; 2\pi \int_{0}^{R(\Theta)} dr \, r^2$$

$$= \int_{-1}^{1} d\cos\Theta \;\; 2\pi \;\; \tfrac{1}{3} \, R^3(\Theta)$$

$$= \int_{-1}^{1} d\cos\Theta \;\; \frac{2\pi}{3} \left[R^3 - 3\,\delta R \, R^2 \, P_2(\cos\Theta) + \cdots \right]$$

$$= \frac{4\pi}{3} R^3 \; - \; 0 \; + \cdots$$

since $\displaystyle\int_{-1}^{1} d\cos\Theta \; P_2(\cos\Theta) = 0$

I claim ~~that~~ this structure will be consistent if we expand to first order in the coefficients of $P_2(\cos\Theta)$. Let's see:

Compute the gravitational forces:

$$\frac{F}{m} = \left(-\frac{\partial}{\partial r}\Phi \, , \; -\frac{1}{r}\frac{\partial}{\partial \Theta}\Phi \right)$$

inside: $\displaystyle\frac{F}{m} = \left(-\frac{4\pi G_N \rho}{3} r - 2Br\,P_2(\cos\Theta) \, , \; -Br \frac{\partial P_2}{\partial \Theta} \right)$

outside: $\displaystyle\frac{F}{m} = \left(-\frac{C}{r^2} - \frac{3D}{r^4} P_2 \, , \; + \frac{D}{r^4}\frac{\partial P_2}{\partial \Theta} \right)$

the $\hat{\Theta}$ components of the force are already of first order

$$\left[\frac{\partial P_2}{\partial \Theta} = -3\sin\Theta\cos\Theta \right]$$

in coefficients of P_2, so we can match them at R

$$- BR \frac{\partial P_2}{\partial \theta} = + \frac{D}{R^4} \frac{\partial P_2}{\partial \theta}$$

which matches for all θ if

$$D = - BR^5$$

To match the radial components of F/m, we need to expand consistently to first order in $P_2(\cos\theta)$:

$$- \frac{4\pi G_N \rho}{3}(R - \delta R\, P_2) - 2BR\, P_2$$

$$= - \frac{C}{(R - \delta R\, P_2)^2} - \frac{3D}{R^4} P_2$$

$$- \frac{4\pi G_N \rho}{3} R + \left(\frac{4\pi G_N \rho}{3} \delta R - 2BR \right) P_2$$

$$= - \frac{C}{R^2} + \left(-2 \frac{C}{R^3} \delta R - \frac{3D}{R^4} \right) P_2$$

this matches for all θ if

$$\frac{4\pi G_N \rho}{3} R = \frac{C}{R^2} \quad \Rightarrow \quad C = \frac{4\pi G_N \rho}{3} R^3$$

$$= (\text{mass of planet}) \checkmark$$

and $\longrightarrow$

$$\frac{4\pi G_N \rho}{3}\,\delta R - 2BR = -2\frac{C}{R^3}\delta R - \frac{3D}{R^4}$$

$$= -\frac{8\pi G_N\rho}{3}\,\delta R + 3BR$$

so

$$4\pi G_N \rho\,\delta R = 5BR \Rightarrow B = \frac{4\pi G_N\rho}{5R}\,\delta R$$

f.) Now go to the rotating frame. This adds a centrifugal term to the potential

$$\Phi_c = -\tfrac{1}{2}\Omega^2 r^2 \sin^2\theta$$

$$= -\tfrac{1}{2}\Omega^2 r^2 (1-\cos^2\theta)$$

$$= +\tfrac{1}{3}\Omega^2 r^2\left[P_2(\cos\theta) - 1\right]$$

The full potential $\Phi + \Phi_c$ must be constant on the surface. Using the inside form:

$$\Phi_{in} + \Phi_c = (\text{const}) + \frac{2\pi G_N\rho}{3}(R - \delta R\,P_2)^2$$

$$+ BR^2 P_2 + \tfrac{1}{3}\Omega^2 R^2 P_2$$

$$= (\text{const})$$

then $$-\frac{4\pi G_N\rho}{3}R\,\delta R + BR^2 + \tfrac{1}{3}\Omega^2 R^2 = 0$$

so that

$$\left(\frac{4}{3} - \frac{4}{5}\right) \pi G_N \rho R \; \delta R = \frac{1}{3} \Omega^2 R^2$$

$$\frac{8}{15} \pi G_N \rho R \, \delta R = \frac{1}{3} \Omega^2 R^2$$

$$\delta R = \frac{5}{8} \frac{\Omega^2 R^2}{\pi G_N \rho R}$$

now

$$R_e - R_p = R(\Theta = \pi) - R(\Theta = 0)$$

$$= \delta R \left[\frac{1}{2} - (-1) \right] = \frac{3}{2} \delta R$$

$$g = \frac{\frac{4}{3}\pi \rho R^3 \cdot G_N}{R^2} = \frac{4}{3} \pi \rho R G_N$$

so

$$R_e - R_p = \frac{15}{16} \frac{\Omega^2 R^2}{\pi G_N \rho R} = \frac{5}{4} \cdot \frac{\Omega^2 R^2}{\frac{4}{3}\pi G_N \rho R}$$

a $$R_e - R_p = \frac{5}{4} \frac{\Omega^2 R^2}{g}$$