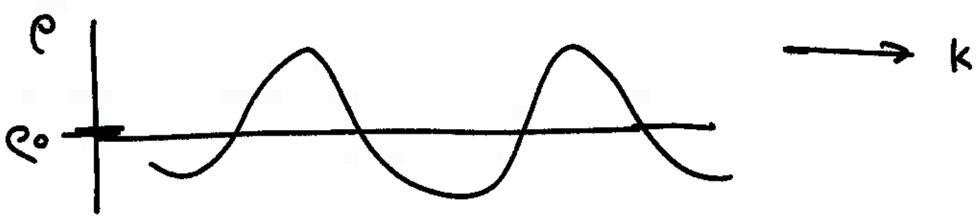
Mar. 4

Shock Waves

In a previous lecture, I discussed some consequences of compressibility in fluids. I derived equations for sound waves, for the more sophisticated types of oscillations that one finds in stratified media, and for the gravitational collapse of a gas cloud. These examples showed that compressibility leads to new physical effects beyond those available in flow at constant density.

However, the effects of compressibility can be even more profound. Flows in a compressible fluid can become catastrophically nonlinear, in such a way that a smooth fluid flow can evolve in a finite time into a state with discontinuities in fluid velocity at a specfic points in space. In this lecture, I will present the theory of these discontinuities, called *shock waves.*

Here is a simple way to understand the possibility of discontinuities arising in a compressible fluid. Begin from a sound wave of finite amplitude.



We have seen already that the speed of sound in a fluid is given by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

We might ask, next, how does the speed of sound depend on density or pressure. What we would expect is that a fluid would become *stiffer* with increased pressure or density. In an adiabatic compression, the temperature increases, and this also increases the speed of sound. We then expect that

$$\left(\frac{\partial^2 p}{\partial \rho^2}\right)_s > 0 \qquad \left(\frac{\partial^2 V}{\partial p^2}\right)_s > 0$$

(with $V = 1/\rho$). The inequalities

$$\left(\frac{\partial p}{\partial \rho}\right)_s > 0 \quad \Leftrightarrow \quad \left(\frac{\partial V}{\partial p}\right)_s < 0$$

are necessary for sound waves to propagate stably in a medium. Landau and Lifshitz stress that the second derivative inequalities have no similar thermodynamic proof, but nevertheless they are general properties of gases. As an example, we can check that for the adiabatic compression of an ideal gas, the relation

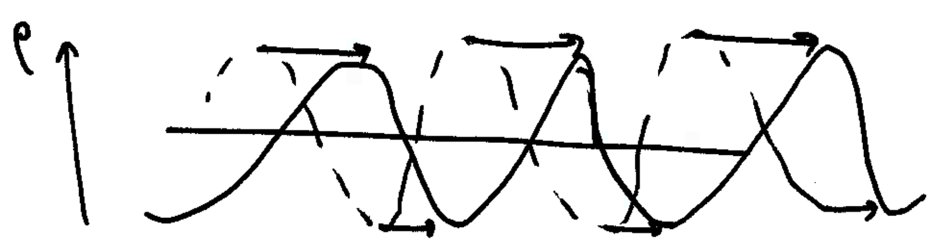
$$p \sim \rho^{\gamma} \quad \text{or} \quad p \sim V^{-\gamma}$$

implies

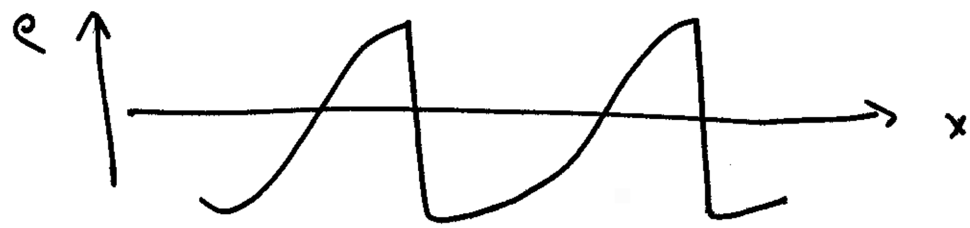
$$\left(\frac{\partial^2 p}{\partial \rho^2}\right)_s = \gamma(\gamma-1)\frac{p}{\rho^2} \qquad \left(\frac{\partial^2 V}{\partial p^2}\right)_s = \frac{\gamma+1}{\gamma^2}\frac{V}{p^2}$$

Since $\gamma > 1$, we see that the second derivative inequalities are satisfied. We will find these inequalities appearing often in the theory of compressible gases.

If a gas is stiffer at higher pressure or density, its speed of sound increases. This has the consequence that, in a pressure wave of finite amplitude, the crests move faster than the troughs. Thus, the waveform drawn above evolves to

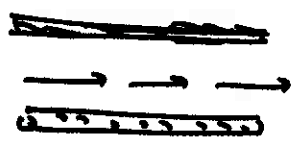


and then to



The pressure and density at each point must have a single value, so the solution develops discontinuities. These discontinuities are the *shock waves.* They can arise in the integration of the Navier-Stokes equations for a compressible gas over a finite time.

I will now develop some formulae describing gas dynamics with finite variations of pressure and density. The simplest situation is that of 1-dimensional gas flow through a pipe.



The relevant variables are

$$V_x = V \qquad \rho \qquad P$$

with

$$\frac{D}{Dt} s = 0$$

The equations governing the fluid flow are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial v}{\partial t} + v \frac{\partial}{\partial x} v = -\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}$$

I will first analyze these equations by looking for a similarity solution that depends on (t, x) only through the combination

$$\xi = \frac{x}{t}$$

Since

$$\frac{\partial}{\partial t}\xi = -\frac{\xi}{t} \qquad \frac{\partial}{\partial x}\xi = \frac{1}{t}$$

the mass conservation equation becomes

$$-\frac{\xi}{t}\rho' + \frac{1}{t}\rho' v + \frac{1}{t}\rho v' = 0$$

$$\text{or} \quad (v-\xi)\rho' + \rho v' = 0$$

where

$$\rho' = \frac{d\rho}{d\xi} \qquad v' = \frac{dv}{d\xi}$$

Similarly, the Navier-Stokes equation becomes

$$(v-\xi)v' + \frac{c^2}{\rho}\rho' = 0$$

These equations are solved trivially by $\rho' = v' = 0$, a constant velocity flow. However, there is a nontrivial solution with $\rho', v' \neq 0$. This condition requires

$$(v-\xi)^2\rho' = c^2\rho'$$

or

$$\xi = v \pm c$$

that is

$$\frac{x}{t} = v \pm c$$

In this similarity solution, the minima and maxima of density and pressure present in the initial condition propagate along the flow at the velocities $v \pm c$, that is, at the local speed of sound relative to the local velocity of the flow.

To analyze the solution further, choose for definiteness

$$\frac{x}{t} = v+c \quad \text{or} \quad v-\xi = -c$$

Then

$$c\rho' = \rho v'$$

or

$$dv = \frac{c}{\rho}\, d\rho$$

with c depending on ρ according to adiabatic compression.

We can now obtain a good qualitative picture of the solution by using thermodynamic inequalities. Take $\partial/\partial x$ of the relation $x/t = v + c$,

$$\frac{1}{t} = \frac{\partial\rho}{\partial x}\,\frac{\partial(v+c)}{\partial\rho} = \frac{\partial\rho}{\partial x}\left(\frac{\partial v}{\partial\rho} + \frac{\partial c}{\partial\rho}\right)$$

$$= \frac{\partial\rho}{\partial x}\left(\frac{c}{\rho} + \frac{dc}{d\rho}\right)$$

or

$$\frac{1}{t} = \frac{\partial\rho}{\partial x}\,\frac{1}{\rho}\,\frac{\partial}{\partial\rho}(c\rho)$$

Now, the quantity ρc is given by

$$\rho c = \rho\sqrt{\left(\frac{dp}{d\rho}\right)_s} = \sqrt{-\left(\frac{dp}{dV}\right)_s} = \left[-\left(\frac{dV}{dp}\right)_s\right]^{-1/2}$$

and so $d/d\rho$ of this quantity is

$$\frac{d}{d\rho}(\rho c) = c^2\,\frac{d}{dp}(\rho c) = c^2\cdot\frac{1}{2}\,\frac{1}{[-(dV/dp)_s]^{3/2}}\left(\frac{\partial^2 V}{\partial p^2}\right)_s$$

$$= \frac{1}{2}c^5\rho^2\left(\frac{\partial^2 V}{\partial p^2}\right)_s$$

In all, we obtain the relation

$$\frac{1}{t} = \frac{\partial\rho}{\partial x}\cdot\frac{1}{\rho}\cdot\frac{1}{2}c^5\rho\cdot\left(\frac{\partial^2 V}{\partial p^2}\right)_s$$

In this formula, all factors are positive except possibly $(\partial\rho/\partial x)$, so that factor must also be positive. Then

$$\frac{\partial \rho}{\partial x} > 0$$

which also implies

$$\frac{\partial p}{\partial x} > 0 \qquad \frac{\partial v}{\partial x} > 0$$

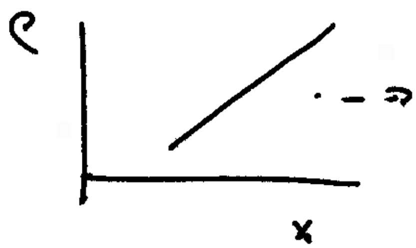
We might also compute

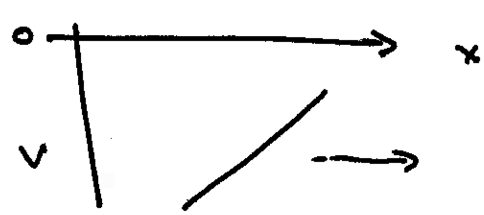
$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = (v-s)\frac{\partial \rho}{\partial x} = -c\frac{\partial \rho}{\partial x}$$

Thus, this quantity is negative, which implies

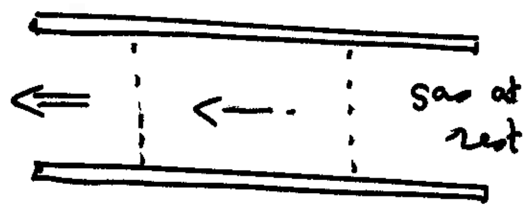
$$\frac{D\rho}{Dt} < 0 \qquad \frac{Dp}{Dt} < 0 \qquad \frac{Dv}{Dt} < 0$$

To satisfy these constraints, we need $v < 0$, an overall flow to the left, as the pattern of the wave moves to the right. The density and velocity then depend on x according to

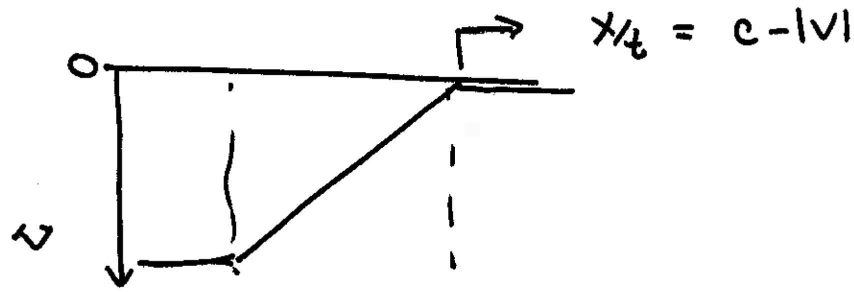




This is a *rarefaction* wave. A possible realization of the wave is a flow



in which the velocity distribution is



We can obtain a more explicit set of formulae by assuming that the equation of state of the gas is that of an ideal gas. Then

$$c = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} \sim \rho^{\frac{\gamma-1}{2}}$$

This implies

$$v = \int \frac{c}{\rho}\, d\rho \sim \int \frac{2}{\gamma-1}\, dc$$

or

$$v = \frac{2}{\gamma-1}\,(c-c_0)$$

where c_0 is the speed of sound in the gas at rest to the right. Note that the density is highest here, and so the speed of sound is greatest. The formula for v does give $v < 0$. The explicit form of the rarefaction wave is then

$$c = c_0 - \frac{\gamma - 1}{2} |v|$$

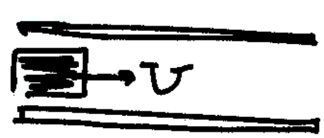
$$\rho = \rho_0 \left[1 - \frac{\gamma - 1}{2} \frac{|v|}{c_0} \right]^{2/\gamma - 1}$$

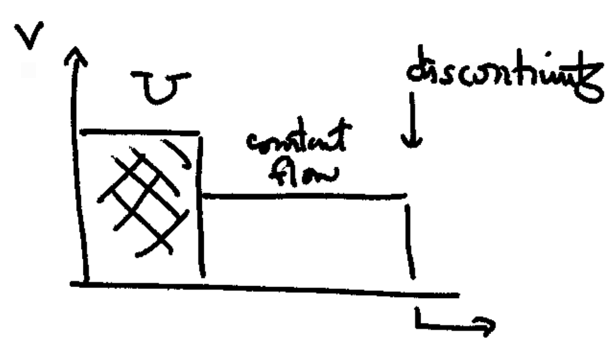
Since $\rho > 0$, $|v|$ cannot become greater than a limiting value

$$v_{lim} = \frac{2c_0}{\gamma - 1}$$

Before this value is reached, we must have the transition to a constant flow indicated in the figure above.

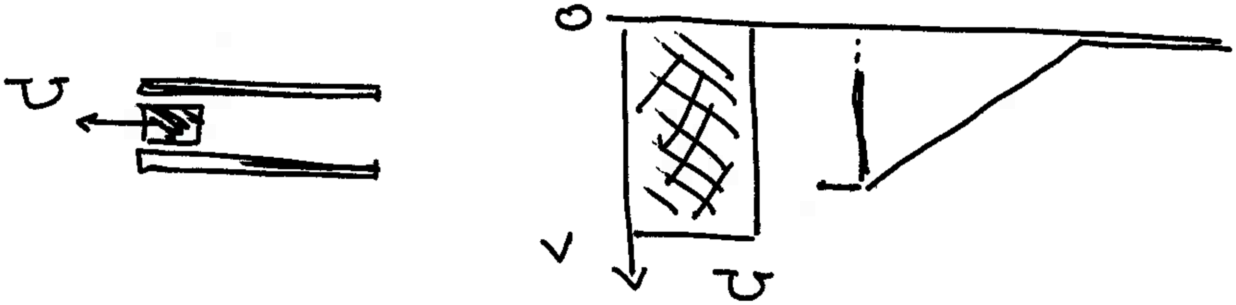
In Landau and Lifshitz, this solution is applied to the problem of gas in a pipe that is being pushed by a piston moving at constant velocity. If will not give the solutions in detail, but I will discuss their qualitative form. For compression, there is no smooth solution available. The gas must then have a region of constant flow, and another region in which the gas is undisturbed.





The endpoint of the region of constant flow propagates to the right at the speed $U + c_0$. The velocity field has a discontinuous jump at this endpoint.

For expansion, we can make use of the nontrivial solution given above. The full solution has a rarefaction wave whose endpoint moves to the right at the speed $U + c_0$. The negative values of v in this wave increase from right to left up to a maximum.



The maximum velocity is U if $|U|$ is less than the limiting value given above. If $|U|$ is greater, the gas cannot catch up to the piston and there is a region of vacuum in front of the piston.

Next, I will consider the same set of 1-dimensional equations from a viewpoint of greater generality. The equations are, again,

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial x} = 0$$

Multiply the first equation by $\pm c/\rho$ and add it to the second equation. This gives

$$\left(\frac{\partial v}{\partial t} \pm \frac{c}{\rho} \frac{\partial \rho}{\partial t}\right) + (v \pm c)\left(\frac{\partial v}{\partial x} \pm \frac{c}{\rho} \frac{\partial \rho}{\partial x}\right) = 0$$

We now define the *Riemann invariants*

$$J_+ = v + \int \frac{c}{\rho} d\rho \qquad J_- = v - \int \frac{c}{\rho} d\rho$$

These quantities obey

$$\left[\frac{\partial}{\partial t} + (v+c) \frac{\partial}{\partial x} \right] J_+ = 0$$

$$\left[\frac{\partial}{\partial t} + (v-c) \frac{\partial}{\partial x} \right] J_- = 0$$

These equations have the form of conservation laws. The quantities J_+, J_- are conserved in convection through the gas at velocities $\pm c$ relative to the overall gas flow. This solution is not completely explicit, because the velocities v and c must be found self-consistently with the values of J_+, J_-. Nevertheless, these equations are a significant step toward both solving the equations and visualizing the gas behavior.

To emphasize that last point, I will now say a bit about equations of the form

$$\left(\frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} \right) Q(x,t) = 0$$

where $v(x)$ is a known velocity field. Often, we want to solve an equation of this type with the initial condition

$$Q(x, t=0) = Q_0(x)$$

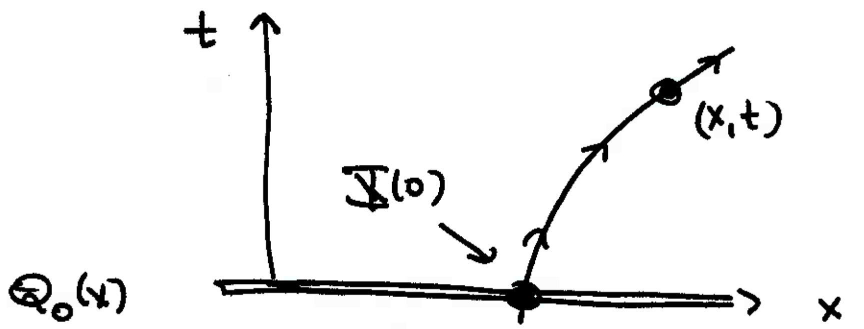
This is done by the following procedure, called the *method of characteristics*: To find Q at the point (x, t), first find the trajectory $X(t)$ such that

$$\frac{d}{dt} X(t) = v(X) \qquad X(t) = x$$

The value of this quantity at $t = 0$ is the original point from which the fluid flowed to reach x at t. This trajectory is called the *characteristic*. The content of the equation is that $Q(x, t)$ is conserved by convection. Thus, the solution is

$$Q(x,t) = Q_0\left(X(0\,;\,x,t)\right)$$

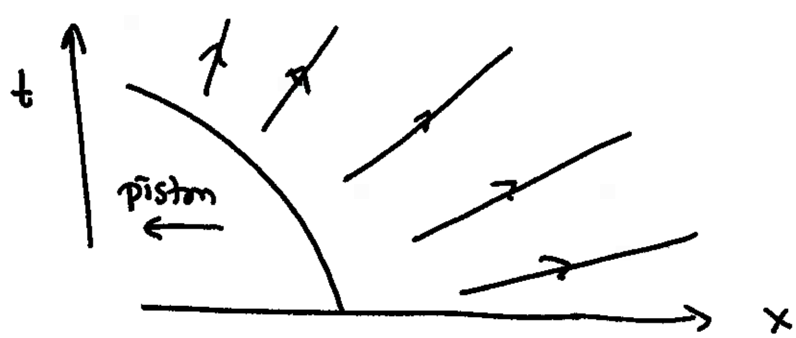
Graphically



In gas flow, as I have already remarked, the solutions are not so simple to find because the velocities must also be found from the solution for the conserved quantities. However, there is a deeper problem, which stems from the fact that the speed of sound increases with density. To see this, we can draw the characteristics for the equation

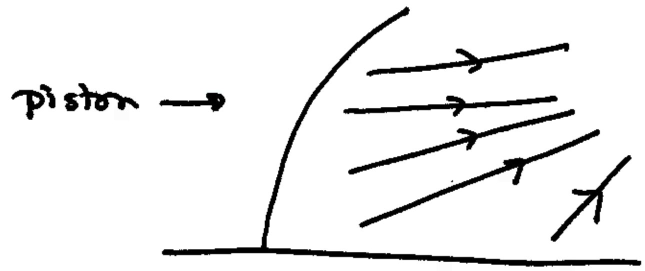
$$\left[\frac{\partial}{\partial t} + (v+c)\frac{\partial}{\partial x}\right] Q = 0$$

for some illustrative situations. For an expanding piston, these trajectories are



The velocities $(v + c)$ decrease with density, so the tranjectories become steeper on the (x, t) plane as the gas is rarefied. The diagram indicates that the solution to the problem is smooth.

However, for a compressing piston, this diagram has the form

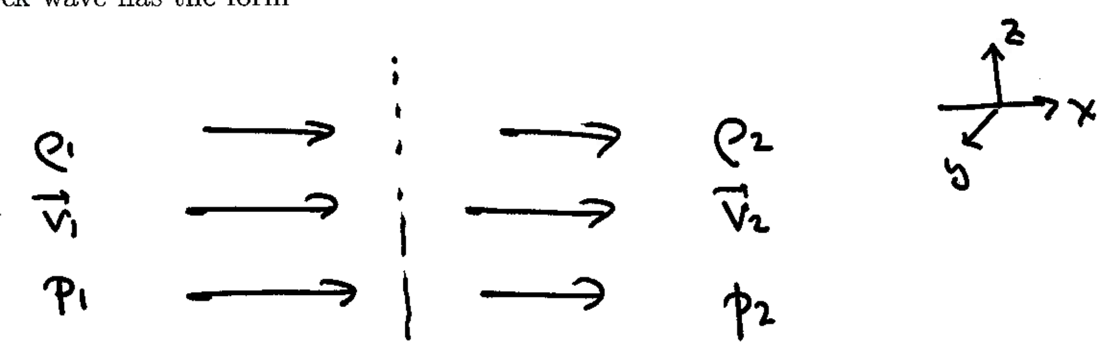


The gas is at higher density at later times, leading to trajectories that are less steep on the (x, t) plane. In this situation, the characteristics can intersect and cross! The region of intersection of characteristics

must contain a discontinuity. This is a somewhat more formal version of the argument given above for the creation of shocks in sound propagation.

It is interesting to make a general analysis of the form of a discontinuity in compressible fluid flow. This general analysis will apply just in the neighborhood of the discontinuity. If there is a discontinuity in a flow, the fluid properties on one side of the flow should be related to those on the other side so that we can integrate the Navier-Stokes equation through the discontinuity. This analysis will give those relations, which are called the *Rankine-Hugoniot conditions.*

Work in the frame of reference in which the discontinuity is it at rest. Orient the $\hat{x}$ axis perpendicular to the discontinuity. Then the fluid flow in the neighborhood of the shock wave has the form



Conserved quantities cannot accumulate on the shock wave. So, each conservation law gives a constraint that the amount of the conserved quantity entering the shock from the left must equal the amount of this quantity exiting the shock to the right. If $\vec{j}$ is the current associated with the conservation law, then,

$$(\vec{j}\cdot\hat{x})_{\text{side } 1} = (\vec{j}\cdot\hat{x})_{\text{side } 2}$$

We can write out these relations for the $\hat{x}$ components of the currents of particle number, $\vec{j}\cdot\hat{x}$,

$$\rho_1 v_{1x} = \rho_2 v_{2x}$$

of momentum, $T^{ik}\hat{x}^k$,

$$p_1 + \rho_1 v_{1x}^2 = p_2 + \rho_2 v_{2x}^2$$

$$\rho_1 v_{1x} v_{1y} = \rho_2 v_{2x} v_{2y}$$

$$\rho_1 v_{1x} v_{1z} = \rho_2 v_{2x} v_{2z}$$

and of energy, $\vec{j}_\epsilon \cdot \hat{x}$,

$$\rho_1 v_{1x} \left(\tfrac{1}{2} v_1^2 + h_1\right) = \rho_2 v_{2x} \left(\tfrac{1}{2} v_2^2 + h_2\right)$$

Notice that the pressure p appears in the diagonal entries of T^{ij} and thus enters the momentum equations. We have five equations, so we can solve for the fluid variables on side 2, given the fluid variables on side 1.

There are two types of solutions. First, it is possible that there is no mass flow cross the discontinuity

$$\rho_1 v_{1x} = 0 = \rho_2 v_{2x}$$

Then, all of the equations are solved if the pressure is continuous,

$$p_1 = p_2$$

This is a *tangential discontinuity* of the type that we encountered in our study of the Kelvin-Helmholtz instability.

This is not a true shock wave, since no energy or momentum is tranferred across the discontinuity.

Alternatively, assume that

$$\rho_1 v_{1x} = \rho_2 v_{2x} \neq 0$$

Then

$$v_{1y} = v_{2y} \qquad v_{1z} = v_{2z}$$

$$p_1 + \rho_1 v_{1x}^2 = p_2 + \rho_2 v_{2x}^2$$

$$\frac{1}{2} v_1^2 + h_1 = \frac{1}{2} v_2^2 + h_2$$

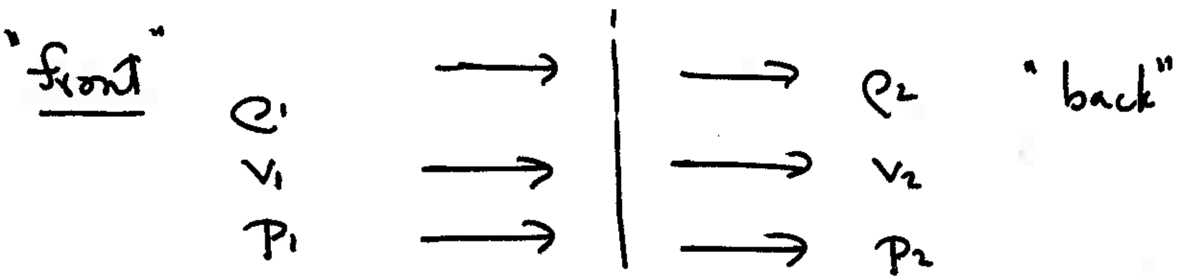
This situation describes a shock wave proper. Notice that, in this case, we can always choose a frame such that

$$v_{1y} = v_{2y} = 0 \qquad v_{1z} = v_{2z} = 0$$

The flow across the shock is then completely 1-dimensional. I will choose the labels 1 and 2 in such a way that

$$j = \rho_1 v_{1x} = \rho_2 v_{2x} > 0$$

as indicated in the figure above. Then mass, momentum, and energy flow through the shock in a well-defined way from a *front* to a *back* side.



I will refer to the side 1 as the front of the shock and side 2 as the back. I will set $v_{1x} = v_1$ and $v_{2x} = v_2$ from here on. Also, let

$$V_1 = \frac{1}{\rho_1} \qquad V_2 = \frac{1}{\rho_2}$$

Now we need to solve the equations

$$j = \rho_1 v_1 = \rho_2 v_2$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2$$

$$\frac{1}{2} v_1^2 + h_1 = \frac{1}{2} v_2^2 + h_2$$

We can rewrite the first equations as

$$v_1 = j V_1 \qquad v_2 = j V_2$$

Then

$$p_1 + j^2 V_1 = p_2 + j^2 V_2$$

This implies

$$j^2 = \frac{p_2 - p_1}{V_1 - V_2}$$

Since $j^2 > 0$, we see from this equation that

$$[p_2 > p_1 \ \underline{\text{and}} \ V_1 > V_2] \ \underline{\text{OR}} \ [p_2 < p_1 \ \underline{\text{and}} \ V_1 < V_2]$$

I will argue later that only the first option here can be realized. A consequence of the relations above is

$$v_1 - v_2 = j(V_1 - V_2) = [(p_2 - p_1)(V_1 - V_2)]^{\frac{1}{2}}$$

Now look at the energy equation

$$h_1 - h_2 + \frac{1}{2} j^2 (V_1^2 - V_2^2) = 0$$

With the equation for j^2 above, we can rewrite this as

$$h_1 - h_2 + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

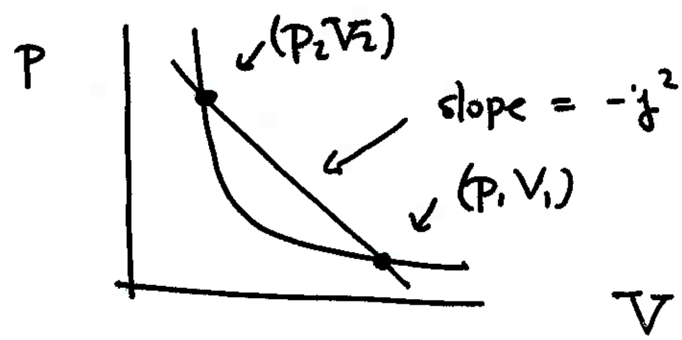
Since

$$u = h - pV$$

this can also be written

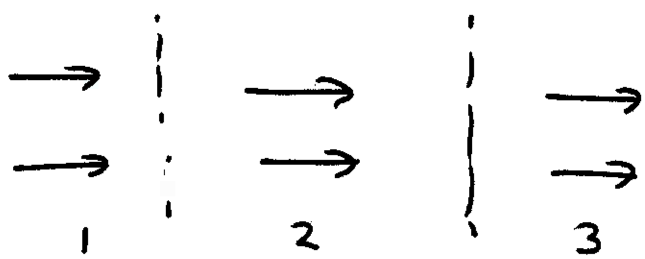
$$u_1 - u_2 + \frac{1}{2}(V_1 - V_2)(p_2 + p_1) = 0$$

These *Rankine-Hugoniot equations* relating the fluid conditions on the front and back of a shock wave are now written in a fairly explicit form, and we can solve them if we know the equation of state. The solution for a given front-side condition (p_1, ρ_1) as a function of j is called the *shock adiabat.* Graphically, it has the form



where, as stated above, the back side always has a higher pressure and density. This curve looks superficially like the constant-s adiabat, or *Poisson adiabat*, $pV^\gamma =$ constant,

However, as we will see in a moment, the shock adiabat is not the equation of a thermodynamic state. If we have a series of shock waves,



it is, in general, *not correct* that 1 and 3 are consistent sides of a single shock wave.

We are now in a position to discuss the ordering of the front and back sides of the shock. To do this, note that each small element of fluid passes across the shock wave from front to back. In this transition, the entropy s of the fluid element cannot decrease. Thus

$$s_2 > s_1$$

I will now work out the implications of this statement of a small jump discontiuity by expanding the thermodynamic variables in powers of the small entropy and pressure differences across the jump.

To begin, expand the difference $h_2 - h_1$ about the state 1,

$$h_2 - h_1 = \left(\frac{\partial h}{\partial s}\right)_p (s_2 - s_1) + \left(\frac{\partial h}{\partial p}\right)_s (p_2 - p_1) + \frac{1}{2}\left(\frac{\partial^2 h}{\partial p^2}\right)_s (p_2 - p_1)^2$$
$$+ \frac{1}{6}\left(\frac{\partial^3 h}{\partial p^3}\right)_s (p_2 - p_1)^3 + \cdots$$

Similarly, $V_2 - V_1$ has the expansion

$$V_2 - V_1 = \left(\frac{\partial V}{\partial p}\right)_s (p_2 - p_1) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial p^2}\right)_s (p_2 - p_1)^2 + \cdots$$

Now we can identify many of the thermodynamic derivatives,

$$\left(\frac{\partial h}{\partial s}\right)_p = T_1 \qquad \left(\frac{\partial h}{\partial p}\right)_s = V_1 \quad \left(\frac{\partial^2 h}{\partial p^2}\right)_s = \left(\frac{\partial V}{\partial p}\right)_s \quad \text{etc.}$$

Put these results into the Rankine-Hugoniot condition

$$h_2 - h_1 = +\frac{1}{2}(V_1 + V_2)(p_2 - p_1)$$

We find

$$T_1(s_2 - s_1) + V_1 (p_2 - p_1) + \frac{1}{2}\left(\frac{\partial V}{\partial p}\right)_s (p_2 - p_1)^2 + \frac{1}{6}\left(\frac{\partial^2 V}{\partial p^2}\right)_s (p_2 - p_1)^3 + \cdots$$
$$= V_1 (p_2 - p_1) + \frac{1}{2}\left(\frac{\partial V}{\partial p}\right)_s (p_2 - p_1)^2 + \frac{1}{2\cdot 2}\left(\frac{\partial^2 V}{\partial p^2}\right)_s (p_2 - p_1)^3 + \cdots$$

There is a nice cancellation, and the final result is

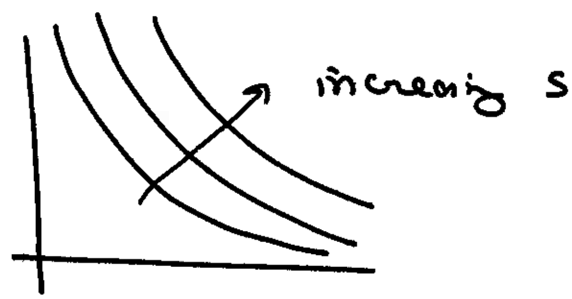
$$s_2 - s_1 = \frac{1}{12 T_1} \left(\frac{\partial^2 V}{\partial p^2}\right)_s (p_2 - p_1)^3 + \dots$$

The coefficient here is the derivative for which I argued at the beginning of the lecture that it should be positive. Thus, for a weak discontinuity,

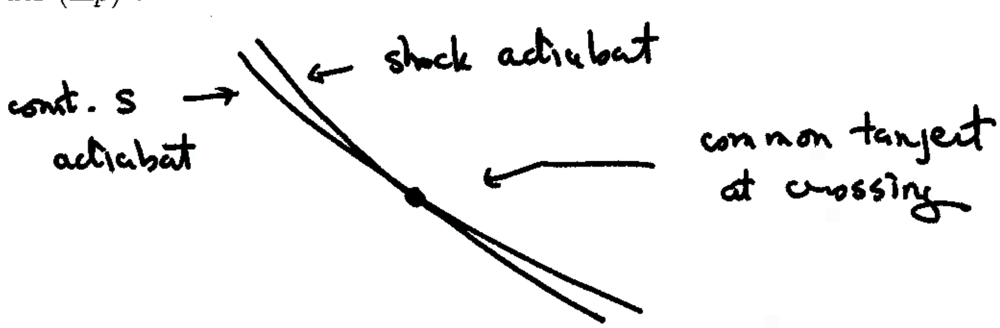
$$s_2 > s_1 \quad \Rightarrow \quad p_2 > p_1$$

In Landau and Lifshitz, you will find a more involved thermodynamic argument that derives this conclusion for a shock wave of any intensity.

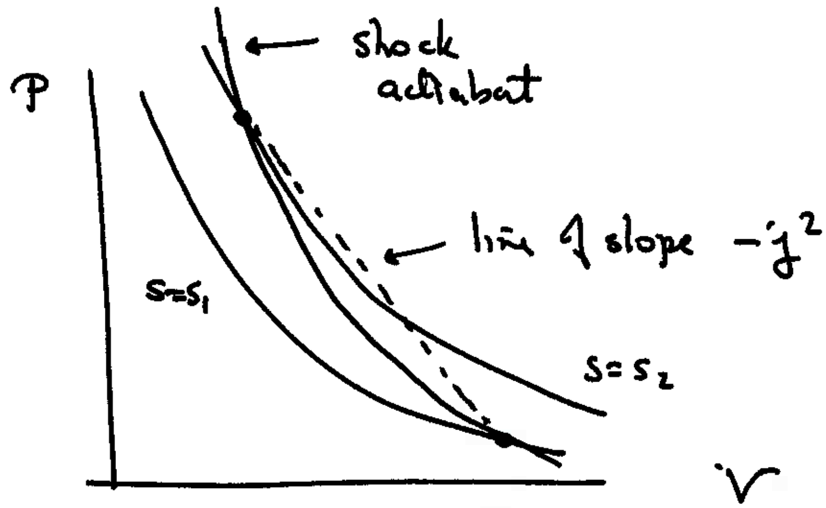
Since p increases with s at constant ρ or V, the Poisson adiabats are stacked on one another as



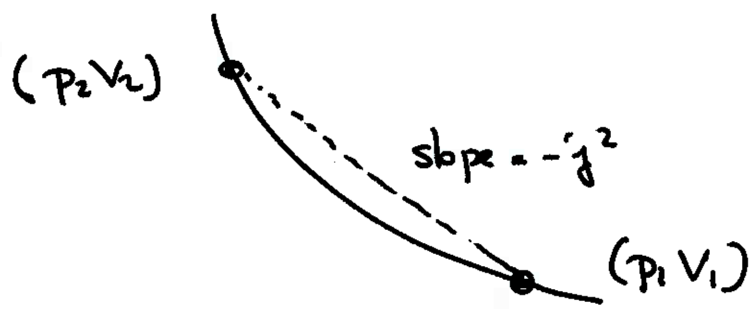
We can draw the shock adiabats on top of these curves. According to the expansion above, the shock adiabat differs from the Poisson adiabat at crossing by terms of order $(\Delta p)^3$.



Then the shock adiabat crosses the Poisson adiabats as follows:



The line that joins the conditions on the two sides of the shock, (p_1, V_1) and (p_2, V_2), has slope $-j^2$. From the figure, we see that the magnitude of this slope is larger than the slope of the adiabats at 1 and smaller at 2.



The speed of sound at (p_1, V_1) is given by

$$c_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,1} = -V_1^2\left(\frac{\partial p}{\partial V}\right)_{s,1} < V_1^2 j^2 = v_1^2$$

Similarly, the speed of sound at (p_2, V_2) is

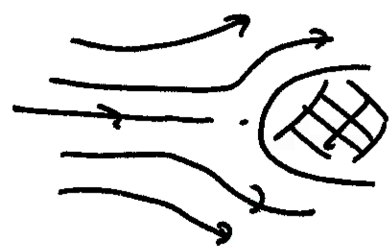
$$c_2^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,2} = -V_2^2\left(\frac{\partial p}{\partial V}\right)_{s,2} > V_2^2 j^2 = v_2^2$$

Thus, finally,

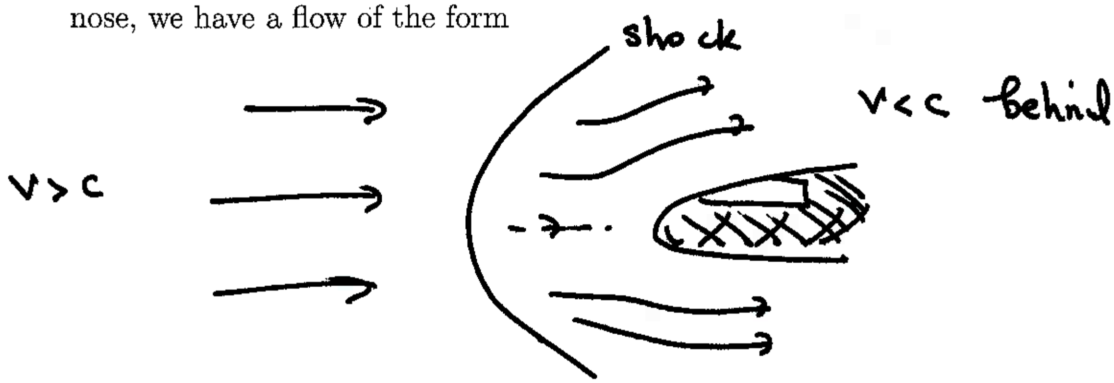
$$v_1 > c_1 \qquad\qquad v_2 < c_2$$

In the frame in which the shock is at rest, the entering flow is *supersonic*, the exiting flow is *subsonic.*

It makes sense that the entering flow should be supersonic. In an ordinary flow, the fluid velocity slows down in front of an obstacle.



This cannot happen in a supersonic flow, because the elements of fluid are travelling too fast to get the news of an obstacle ahead. This means that the response of the fluid to the obstacle must be sudden, occurring at a spatial discontinuity. On the other hand, it is a little surprising that the flow on the back side is subsonic. This has interesting implications. For example, a jet airplane flying faster than c has a shock wave in front of it that travels at the speed of the jet. In the frame of the jet, the flow between the shock and the jet is subsonic. Then, if the jet has a rounded nose, we have a flow of the form



The shock wave stands off a bit in front of the airplane. The drag on the jet is reduced if the shock can emanate from the tip of the nose. To arrange this, jets have pointed noses.