Physics 211 – Problem Set # 7

(due Thursday, March 11)

1. Here is a very oversimplified model of a hurricane: Start with a layer of air of fixed height H and density ρ, with a two-dimensional flow $\vec{v}$. A convective flow (outside the scope of this problem) sets up a vertical motion that carries a mass Q per unit area (g/cm^2) up out of this layer. Q is constant in circle of radius L and zero outside it. The flow is subject to a Coriolis force with $\vec{f} = 2\Omega \cos\theta \hat{z}$, where $(\pi - \theta)$ is latitude. Consider the hurricane as local and treat $\vec{f}$ as a constant. Look for a rotationally symmetric steady flow.

   (a) Write, for the two-dimensional flow,

   $$\vec{v} = v_r(r)\hat{r} + v_\phi(r)\hat{\varphi}$$

   Using the equation of conservation of mass, solve for $v_r(r)$.

   (b) Write the Navier-Stokes equation with a damping term (*e.g.*, from the flow of wind across the ground,

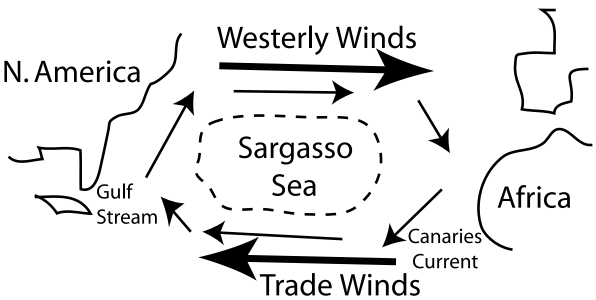
   $$\frac{\partial}{\partial t}\vec{v} + \vec{v} \cdot \vec{\nabla}\vec{v} = -\vec{f} \times \vec{v} - \vec{\nabla}\phi - a\vec{v}$$

   Assume a steady flow, and ignore the nonlinear convective term. (I said this was oversimplified.) If you take the curl of this equation, φ will drop out, then solve for $v_\phi(r)$.

   (c) Solve the for effective pressure φ.

   (d) Make a sketch of the final flow and pressure pattern.

2. In the north Atlantic Ocean, there is the pattern of winds and ocean currents shown in the figure. Westerly winds blow from west to east at 40 degrees latitude. Trade winds blow from east to west at 20 degrees latitude. In between, at around 30 degrees latitude, is the Sargasso Sea, a 1.5-meter high gyre (raised hump of water). The gyre is created by ocean surface currents, extending down to a depth of only about 30 meters, that flow northward from the trade-wind region and southward from the westerly wind region. A deep ocean current, extending from the surface down to near the bottom, circulates around the Sargasso Sea gyre in a clockwise manner. This current goes under different names in different regions of the ocean: gulf stream, west wind drift,



Canaries current, and north equatorial current. Explain both qualitatively and semi-quantitatively how the winds are ultimately responsible for all of these features of the ocean. [B&T 15.10]

More specifically,

(a) The surface currents are explained in terms of an Ekman layer at the top of the Ocean. Compute the depth of the Ekman layer, assuming an otherwise static ocean and the viscosity of water. The actual depth is about 30 meters. This results from an effective 'eddy viscosity' due to turbulent mixing at the surface of the ocean. What effective value of ν is needed?

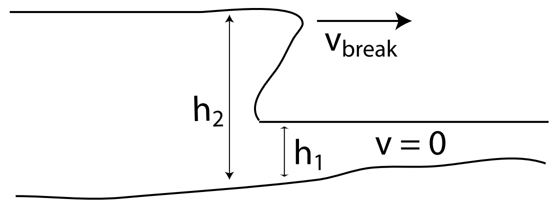
(b) Explain how the Ekman layer produces the gyre and show that 1.5 meters is a reasonable value for its height.

(c) Explain the deep ocean current in terms of a geostrophic flow, and estimate the speed of this current.

(d) If there were no continents on Earth, but only an ocean of uniform depth, what would be the flow pattern of this deep current—its directions of motion at various points around the Earth, and its speeds? The continents (North America, Europe, and Africa) must be responsible for the deviation of the actual current from this continent-free flow pattern. How do you think the continents give rise to the altered flow pattern?

3. The theory that we developed for a shallow layer of incompressible fluid with mainly two-dimensional flow gives interesting illustrations of some of the topics in our discussion of compressible gas dynamics. In this problem, we will analyze incompressible flow in a thin layer. Assume that the flow is effectively one-dimensional, that is, depending on x but independent of y, with the fluid layer thin in the vertical z direction. [B&T 15.7, 16.6].

(a) Consider the flow of water in a channel after a dam breaks. The bottom of the channel is at a constant level. The height of the fluid above this level is $h(x)$.



The fluid velocity in the x direction is $v(x)$. Show that the flow is described by the equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0 \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

(b) Consider these equations with the initial condition: $v(x) = 0$, $h(x) = h_0$ for $x < 0$, $h(x) = 0$ (no water) for $x > 0$. From the parameters of the problem, we can make one characteristic velocity: $v_0 = (gh_0)^{1/2}$. Look for a self-similar solution of the form

$$h = h_0\overline{h}(\xi) \qquad v = v_0\overline{v}(\xi) \qquad \xi = \frac{x/t}{v_0}$$

This converts the pde's above into a set of two odes that can be solved.

(c) Now study the equations above in more generality. Find two Riemann invariants that obey conservation laws that they are constant along characteristics.

(d) Analyze the characteristics and show that shallow-water waves steepen and form shock wave solutions ('hydraulic jumps').

(e) Present a second derivation of the solution to part (a) using the Riemann invariants.

4. Use shallow-water theory to study a breaking ocean wave. [B&T 16.7]

(a) Find the boundary conditions at a hydraulic jump, a discontinuity in the wave height, from the constraints of mass and momentum conservation. Work in the frame in which the jump is at rest. The conservation laws should relate the height and water speed behind the jump, h_2 and v_2, to the height and water speed in front, h_1 and v_1, in this frame. The momentum conservation equation will involve the pressure, which is related to the height.

(b) Show that the upstream speed v_1 in the frame of part (a) is greater than the speed $\sqrt{gh_1}$ of small-amplitude shallow water gravity waves. That is, the upstream flow is 'supersonic'. Show, similarly, that the downstream flow speed is less than $\sqrt{gh_2}$, that is, 'subsonic'.

(c) We normally view a breaking ocean wave in the rest frame of the quiescent upstream water. Compute the speed of the breaking wave in this frame using the hydraulic jump equations, and show that

$$v_{break} = \left[\frac{g(h_1 + h_2)h_2}{2h_1}\right]^{1/2}$$

where h_1 and h_2 are the heights of the water in front of and behind the breaking wave.