Physics 211 – Problem Set # 6

(due Thursday, February 25)

1. Consider a thin jet of fluid emerging from a slit into a volume of fluid at rest and setting up a two-dimensional high Reynolds number flow. Let $\hat{x}$ be the direction of the jet and $\hat{y}$ be the orthogonal direction normal to the slit. Ignore gravity. If the jet is very narrow, the gradient of pressure transverse to the jet will not intergrate to a significant pressure difference, so ignore the pressure term in the Navier-Stokes equation. The leading terms for a narrow jet are then

$$v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x = \nu \frac{\partial^2}{\partial y^2} v_x$$

   (a) Characterize the jet by saying that the slit emits a fixed amount of momentum F per unit length in $\hat{z}$ per unit time. Argue from this that

$$\int_{-\infty}^{\infty} v_x^2 dy$$

   is independent of the position x along the jet.

   (b) Write the approximate Navier-Stokes equation above as an equation for the stream function ψ that gives $v_x = \partial\psi/\partial y$, $v_y = -\partial\psi/\partial x$. Look for a scaling solution to this equation, of the form

$$\psi = Ax^p f(\eta) \qquad \eta = y/x^q$$

   What values of p and q are required?

   (c) Find the differential equation for $f(\eta)$. This should be solved with the boundary condition that $f'(\eta) \to 0$ as $\eta \to \pm\infty$.

   (d) Show that

$$f = B \tanh(\alpha\eta)$$

   solves this equation for suitable B, α.

   (e) Relate the parameter α to F in part (a).

2. Consider Poiseuille flow with central velocity V_0 in a pipe that is maintained at a constant temperature T_0. Because of the fluid's viscosity, heat will be produced in the pipe. This heat will diffuse and the system will reach an equilibrium temperature distribution that is independent of the position z along the pipe.

   (a) Compute the source term in the equation for temperature diffusion that is due to the viscous heating.

(b) Solve this diffusion equation to find the equilibrium distribution $T(r)$.

3. Work out the scaling law, similar to that found in Problem 1, for a thermal plume. Consider a hot knife-edge setting up a two-dimensional flow. Characterize the thermal plume as a narrow upward jet of fluid surrounded by a region of fluid at rest. Gravity will be relevant, but you can ignore viscosity. Let $\hat{z}$ be the direction of the jet and $\hat{y}$ be the orthogonal direction normal to the slit. If the jet is narrow, we can ignore the difference in pressure at fixed z between the pressure in the jet and the pressure in the equilibrium situation far away. [B&T 17.6].

   (a) Characterize the jet by saying that the knife-edge emits a fixed amount of heat Q per unit length in $\hat{x}$ per unit time. Argue from this that

   $$Q = \rho c_p \int_{-\infty}^{\infty} \Delta T \, v_z dy$$

   is independent of the position z along the jet.

   (b) The bouyancy of the fluid leads to a force that increases the mechanical energy in the flow. Find a differential equation that relates $v_z(z)$ to $\Delta T(z)$.

   (c) The width $\delta(z)$ of the plume is determined by diffusion: $(\delta(z))^2 = \chi t$, wher t is the time the fluid takes to reach a height z. Combine these three ingredients, stir, and show that $\delta(z) \sim z^{2/5}$.

   (d) Find expressions for δ, v_z, and ΔT that include the correct power law dependences on z, χ, the density expansion coefficient α, and $Q/\rho c_p$.

   (e) Repeat this exercise for a thermal plume from a point source of heat, which gives a narrow jet in a three dimensions.