Physics 211 – Problem Set # 5

(due Thursday, Feburary 11)

1. For small-wavelength oscillations at an interface, surface tension may come into play. Add a surface tension γ to the analysis of the Kelvin-Helmholtz instability.

   (a) The presence of γ adds a term to the energy of the system. Consider a wave on the interface whose profile is independent of y, that is, with surface position $z = \zeta(x, t)$. A surface tension adds an energy γ per unit area of the interface. Show that this gives a contribution to the energy of the system

   $$\Delta E = \int d^2 x_\perp \; \gamma \left(1 + \frac{1}{2} (\nabla_x \zeta)^2 + \cdots \right) \; .$$

   Show that this changes the boundary condition for the pressure to

   $$p_2 - p_1 = \gamma \nabla_x^2 \zeta$$

   [The formula is also correct for a general surface wave which has surface position $z = \zeta(x, y, t)$, if we replace ∇_x with $\vec{\nabla} = (\nabla_x, \nabla_y)$. Can you prove it?]

   (b) Work out the dispersion relation for pure surface tension waves ($\rho_1 = \rho_2$, $V_1 = V_2 = 0$ in the Kelvin-Helmholtz problem). Show that

   $$\omega^2 = \frac{\gamma k^3}{2\rho}$$

   (c) Work out the dispersion relation in the case $V_1 = 0$ with general $\rho_1 > \rho_2$, and general V_2. Show that the situation is absolutely stable if V_2 is sufficiently small. Find the minimum value of V_2 for which there is an instability.

   (d) Plug in the numbers for wind-generated waves on the sea. Use $\rho_1 = 10^3$ kg/m^3, $\rho_2 = 1.25$ kg/m^3, $\gamma = 0.074$ N/m. How fast must the wind be blowing to generate waves on a flat surface?

2. Consider a jet of fluid in air, ignoring gravity and viscosity but considering surface tension. Within a cylinder of radius a, oriented in the $\hat{z}$ direction, the fluid is moving with $\vec{v} = V_0 \hat{z}$. We can boost to the frame of the fluid, and in this frame it is obvious that the situation is rendered unstable by surface tension. Analyze the instability. This is the physics that governs the breakup of a water jet into droplets.

   (a) Show that the unperturbed situation has $\vec{v} = 0$, $p = p_0 + \gamma/a$ inside the cylinder, for atmospheric pressure p_0. Write $r = \zeta(\phi, z)$ as the position of the surface; then the unperturbed situation is $\zeta = a$.

(b) Let $\delta\zeta$ be the deviation from the unperturbed position and let δp be the deviation in the pressure. Show that the boundary condition on the pressure, to linear order, is

$$\delta p(r = a) = -\gamma(\frac{\delta\zeta}{a^2} + \frac{\partial^2 \delta\zeta}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2 \delta\zeta}{\partial \phi^2})$$

and that the boundary condition on the velocity is

$$v_r = \frac{\partial \delta\zeta}{\partial t}$$

Find the linearized partial differential equations for $\vec{v}$ and δp.

(c) Assume potential flow. Show that the pressure δp satisfies the Laplace equation $\nabla^2 \delta p = 0$. Look for a solution of the form

$$(\vec{v}, \delta p, \delta\zeta) = (\vec{v}(r), p(r), \zeta(r))e^{-i\omega t + ikz + in\phi}$$

Show that the Laplace equation takes the following form on $p(r)$:

$$[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + (k^2 + \frac{n^2}{r^2})]p = 0$$

(d) The equation in part (d) is a modified Bessel equation, with solution

$$p = AI_n(kr)$$

Using the boundary conditions, find the dispersion relation $\omega(k)$.

(e) Show that the perturbations are stable for $|n| > 0$, and for $n = 0$ if k is sufficiently large. What is the value of k below which there is an instability?

3. Consider a cylindrical column of fluid of radius a in bulk rotation $v_\phi = \Omega r$. This is a Couette system with $\Omega_1 = \Omega_2 = \Omega$, $R_1 = 0$, $R_2 = a$. Ignoring viscosity (and using the analysis done in class), find the oscillation frequencies for axisymmetric perturbations. Your answer will involve roots of Bessel functions.