Physics 211 – Problem Set # 4

(due Thursday, Feburary 4)

1. For fluid flow in a pipe, we wrote the formulae for Poiseuille flow assuming a constant pressure gradient dp/dz. However, this is probably not quite correct. Near the front end (intake) of the pipe, the velocity distribution is probably closer to uniform across the cross section, $\vec{v} = V_{in}\hat{z}$, with a thin boundary layer at the edge. As we go down the pipe, the boundary layer becomes thicker. Eventually, its size is of the order of the radius R of the pipe and the velocity distribution becomes Poiseuille flow with central velocity V_0, as described in class. [B&T 13.9]

   (a) Relate V_{in} to V_0.

   (b) Compute the force on the pipe for values of z close to the intake, using the formula for a thin boundary layer.

   (c) At what value of z does the thickness of the boundary layer become of the order of R? (Assume that the pipe is much longer than this value.)

   (d) Estimate the mass flow through the pipe. In class, we found

   $$Q = \frac{\pi}{8}\frac{R^4}{\nu}\frac{p - p_0}{\ell}$$

   where p is the pressure at the intake and ℓ is the total length of the pipe. We might correct this formula for the effect described in this problem by replacing ℓ by an effective length of the pipe ℓ_{eff}. Is ℓ_{eff} larger or smaller than ℓ? Estimate $(\ell_{eff} - \ell)$.

2. Here is a nice mathematical exercise in 'matched asymptotic expansions': Consider the theory for small ϵ of the differential equation

   $$(1+\epsilon)x^2\frac{dy}{dx} = \epsilon[(1-\epsilon)xy^2 - (1+\epsilon)x + y^3 + 2\epsilon y^2]$$

   with the boundary condition $y(1) = 1$.

   (a) By straightforward perturbation theory, using successive approximations in ϵ, compute the solution $y(x)$ through terms of the order of ϵ^2.

   (b) Extrapolate to $x \to 0$. Where does the perturbation theory break down?

   (c) Define an inner expansion that contains the leading terms in the problemmatical region.

(d) Compute the leading term in that region, and match its coefficient to the series computed in part (a).

(e) Compute the second term in the inner series, and formulate a uniformly valid second approximation.

3. Consider a two-dimensional flow at high Reynolds number past a wedge of angle $\beta\pi$. More precisely, consider a flow that becomes parallel to $\hat{x}$ far to the left, but, at $\vec{x} = 0$, encounters a wall that runs along the lines $(x, y) = (r\cos\beta\frac{\pi}{2}, \pm r\sin\beta\frac{\pi}{2})$.

   (a) Solve for the potential flow with this geometry. Find the power law $v_r \sim r^\gamma$ for the potential flow velocity near $r = 0$.

   (b) Following the method described in class, work out the theory of the boundary layer. You should find that the boundary layer is described by a scaling solution for a function $f(w)$ satisfying the differential equation

   $$f''' + ff'' + \beta(1 - f'^2) = 0$$

   Give the boundary conditions needed to solve for $f(w)$ numerically.

   (c) For a flat plate, the force per unit area near the tip of the plate diverged as $x^{-1/2}$. What is the situation here?