Physics 211 – Problem Set # 3

(due Thursday, January 28)

These questions deal with viscous flows in cylindrical or spherical coordinates. Please deal carefully with operators $\vec{\nabla}$ acting on $\vec{v}$ in curvilinear coordinates. One way that is tedious but foolproof is to use explicit formula, *e.g.*, for cylindrical coordinates,

$$\vec{v} = v_r \hat{r} + v_\phi \hat{\phi} + v_z \hat{z}$$

together with

$$\frac{\partial}{\partial r}\hat{r} = \frac{\partial}{\partial r}\hat{\phi} = 0 \qquad \frac{\partial}{\partial \phi}\hat{r} = \hat{\phi} \qquad \frac{\partial}{\partial \phi}\hat{\phi} = -\hat{r}$$

You need to take similar care in evaluating the components of the stress tensor $\nabla^i v^j$.

These are standard problems that everyone should work through. The solutions to these problems are given in Landau and Lifshitz. But, the analyses are straightforward enough that you can solve them using only the information given here.

1. Consider the steady flow between two long cylinders of radii R_1 and R_2, $R_2 > R_1$ rotating about their axes with angular velocities Ω_1, Ω_2 (*Couette flow*). Look for a solution of the form

$$\vec{v} = v(r)\hat{\phi} \qquad p = p(r)$$

   (a) Write out the Navier-Stokes equations and find differential equations for $v(r)$ and $p(r)$. You should find that these equations have relatively simple solutions, *i.e.*

$$v(r) = ar + \frac{b}{r}$$

   (b) Fix the constants a and b from the boundary conditions. Determine the pressure $p(r)$.

   (c) Compute the friction forces that the fluid exerts on the cylinders, and compute the torque on each cylinder. Show that the total torque on the fluid is zero (as must be the case).

2. Two parallel circular disks of radius R are placed close to one another (with separation d), with the space between them filled with fluid. The disks are made to approach one another at a speed u, squeezing the fluid out at the edges. Assume that $d \ll R$. Let z be the coordinate perperdicular to the plates. Assume that the flow is slow, so that you can neglect the nonlinear term $(\vec{v} \cdot \nabla)\vec{v}$, and symmetrical about the axis.

   (a) Find v_z in terms of v_r from the equation $\vec{\nabla} \cdot \vec{v} = 0$.

(b) Write the Navier-Stokes equation for steady flow. Show that $|dp/dz| \ll |dp/dr|$. Thus, ignore dp/dz from here on.

(c) Find the dependence of v_r on z, assuming that dp/dr changes slowly on the scale of d.

(d) Using the relation from part (a), the result of part (b), and the boundary conditions $v_z = 0$ at $z = 0$ and $v_z = -u$ at $z = d$, find an equation for $p(r)$. Solve this equation.

(e) Compute the upward force on the top disk.

3. Consider a sphere of radius R rotating about its axis at the angular velocity $\vec{\Omega}$ in a large volume of fluid. Assume that the Reynolds number is low, so that you can neglect the $(\vec{v} \cdot \nabla)\vec{v}$ term in the Navier-Stokes equation.

   (a) Argue that the pressure is constant in this situation.

   (b) Show that the simple formula

   $$\vec{v} = \vec{\nabla} \times (f(r)\vec{\Omega})$$

   solves the Navier-Stokes equations if $\nabla^2 f = 0$.

   (c) Choosing a monopole form of $f(r)$, we obtain

   $$\vec{v} = \frac{R^3}{r^3}\vec{\Omega} \times \vec{r}$$

   Show that this satisfies the boundary condition.

   (d) Compute the frictional force on a unit area of the sphere, and find the total torque on the sphere. Show that

   $$\vec{\tau} = -8\pi\eta R^3 \Omega$$

   (e) Now consider a sphere in oscillating rotation $\vec{\Omega}(t) = \mathrm{Re}[\vec{\Omega}_0 e^{-i\omega t}]$. Look for a solution of the form of part (a), inside Re[]. Show that

   $$(\nabla^2 + k^2) f = 0 \ , \quad \text{with} \quad k^2 = i\omega/\nu$$

   (f) A simple solution of this equation is

   $$f = a\frac{e^{ikr}}{r}$$

   Compute $\vec{v}$, show that the boundary condition can be satisfied, and determine the (complex) coefficient a.

   (g) Show that this solution reduces to the previous one as $\omega \to 0$.

   (h) Find leading term for the torque on the sphere $\vec{\tau}(t)$ in the limit $\omega \to \infty$.