Physics 211 – Problem Set # 2

(due Thursday, January 21)

1. In steady flow of an ideal fluid, the Bernoulli function B is conserved along streamlines: $\vec{v} \cdot \vec{\nabla} B = 0$. Show that the variation of B across streamlines is given by $\vec{\nabla} B = T\vec{\nabla} s + \vec{v} \times \vec{\omega}$. This is *Crocco's Theorem.* Apply this to a tornado. In the core of the tornado, the velocity is zero. The velocity is also zero well outside. Is the pressure higher or lower in the interior? Put in some numbers; what pressures are achieved in a tornado that tears a house apart? (B&T 12.10)

2. Suppose that a spherical bubble has been created in water, and analyze its collapse. Assume that the pressure is zero inside the bubble and is p_0 at infinity. Assume that the collapse is spherically symmetric. (B&T 12.13)

   (a) Let the radius of the bubble be $R(t)$, with initial value R_0. Show that the fluid velocity outside the bubble at a distance r is $v = F(t)/r^2$. Using the Euler equation, show that
   $$\frac{1}{r^2}\frac{dF}{dt} + v\frac{dv}{dr} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

   (b) Integrate this equation outward to infinity and show that
   $$-\frac{1}{R(t)}\frac{dF}{dt} + \frac{1}{2}v^2(R) = \frac{p_0}{\rho}$$

   (c) From this, show that the bubble collapses with the speed
   $$v(R) = \left(\frac{2p_0}{3\rho}\right)^{1/2}\left[\left(\frac{R_0}{R}\right)^3 - 1\right]^{1/2}$$
   Note the singularity as $R \to 0$!

3. In class, we derived the flow around a cylinder using conformal mapping and the Joukowski transformation
   $$w = \frac{1}{2}(z + \frac{1}{z})$$
   I have scaled the formula so that a circle of radius 1 in the z plane is transformed into a slit from -1 to 1 in the w plane. Work out some further properties of this transformation.

   (a) Show that the Joukowski transformation can be expressed as
   $$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2$$

(b) Show that the Joukowski transformation doubles angles at the points $z = \pm 1$.

(c) Show that the Joukowsky transformation maps circles $|z| = \rho$ into a family of confocal ellipses in the w plane. The parameters ρ and $1/\rho$ give the same ellipse, and $\rho = 1$ give the degenerate case of a slit along the real axis.

(d) Show that the Joukowsky transformation maps half-lines $\arg z = \phi$ into a family of confocal hyperbolas orthogonal to these ellipses.

(e) If λ is real, the equation $|z - i\lambda| = \sqrt{\lambda^2 + 1}$ is the equation of a circle in the z plane that goes through $z = \pm 1$. Show that the two pieces of the circle, above and below the real axis, are mapped to the same curve in the w plane. Sketch that curve.

(f) If k is real and > 0, consider the circle $|z + k| = k + 1$. Find the shape of the image of this figure in the w plane. Show that this shape is a teardrop with a cusp with horizontal tangent at $z = 1$.

(g) Finally, consider the circle

$$|z + i\lambda + k(i\lambda + 1)| = (1 + k)\sqrt{\lambda^2 + 1}$$

Sketch the shape of the image in the w plane. Does this figure remind you of anything?

(h) Sketch the potential flow around the figures created in parts (f) and (g). Show that there is a stagnation point at the front edge of the figure. How does $\vec{v}(x)$ behave near the cusp? Is this physical?

4. Here is a bit more about gravity waves on the surface of an incompressible fluid:

(a) In class, we wrote the solution for gravity waves in a situation in which the depth of the fluid is much larger than the wavelength. Using these formulae compute the kinetic energy of the wave per unit area, $\int d^2x(\frac{1}{2}\rho v^2)$ and the potential energy of the wave per unit area, $\int d^2x(\rho\Phi)$ averaged over a cycle.

(b) Solve for the motion of a gravity wave in the case the the fluid has a finite depth ℓ.

(c) Find the kinetic and potential energies in this more general case.

(d) Take the limit of these formulae for shallow water, $k\ell \ll 1$.

(e) A deep water wave approaches a beach. Adiabatically, the water becomes more shallow until $k\ell \ll 1$. What happens?