Feb. 25

Rotating Fluids

The next modification of a simple structureless fluid that I will consider is the addition of overall rotation to fluid motion. The simplest systems that we might consider as examples are fluids in bulk rotation in cylindrical containers. However, the theory of rotating fluids has important applications to the motion of fluids on the rotating earth and on other planets and stars. In this lecture, I will confine myself to simple laboratory situations; in the next lecture, I will describe some applications to the theory of planetary fluid dynamics. For simplicity, I will confine my attention in these lectures to incompressible fluids (constant ρ).

We can describe a fluid in bulk rotation by working in a rotating coordinate system. As you know from your mechanics courses, physics in a rotating coordinate system includes fictitious forces. Here is a derivation of these forces. Consider the description of motion in a frame rotating by

$$\vec{\Omega}$$

with respect to an intertial frame. The relation of the frames is described by a rotation matrix $R_{jk}(t)$ whose time evolution satisfies the differential equation

$$\frac{d}{dt} R_{jk}(t) V_k = R_{jk} (\vec{\Omega} \times \vec{V})_k$$

for every vector V^i. The content of this equation is that $R(t)$ absorbs the rotation of vectors $\vec{V}$ by $\vec{\Omega}$.

Now let $\vec{x}_i(t)$ be the position of a particle with respect to the inertial frame and let $\vec{x}_r(t)$ be the position of this particle with respect to the rotating frame. These are related by

$$\vec{x}_i(t) = R_{jk}(t)\, x_r^k(t)$$

Then

$$\frac{dx_i}{dt} = R(t)\left[\frac{d\vec{x}_r}{dt} + \vec{\Omega}\times\vec{x}_r\right]$$

$$\frac{d^2x_i}{dt} = R(t)\left[\frac{d^2\vec{x}_r}{dt^2} + 2\vec{\Omega}\times\frac{d\vec{x}_r}{dt} + \vec{\Omega}\times\vec{\Omega}\times\vec{x}_r\right]$$

The last term can be simplified

$$\vec{\Omega}\times\vec{\Omega}\times\vec{x}_r = \vec{\Omega}\;\vec{\Omega}\cdot\vec{x}_r - \Omega^2\vec{x}_r = -\Omega^2(\vec{x}_r)_\perp$$

The equation of motion in the inertial frame is

$$M\frac{d^2x_i}{dt^2} = F_i(t)$$

where $\vec{F}_i(t)$ is the force on the object with respect to the inertial frame. The same force with respect to the rotating frame is given by

$$F_i(t) = R(t)\, F_r(t)$$

Then the equation of motion for $\vec{x}_r(t)$ becomes

$$\frac{d^2 x_r}{dt^2} = \frac{\vec{F}_r}{M} + \Omega^2 (\vec{x}_r)_\perp - 2\vec{\Omega} \times \frac{d\vec{x}_r}{dt}$$

There are two fictitious forces. The first is the *centrifugal force.* This is described by a potential

$$\Omega^2 (\vec{x}_r)_\perp = -\vec{\nabla} \Phi_c \qquad \Phi_c = -\frac{\Omega^2 (x_r)_\perp^2}{2}$$

The second is the *Coriolis force.*

The Navier-Stokes equation in the rotating frame takes the form

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \Phi - \vec{\nabla} \Phi_c - 2\vec{\Omega} \times \vec{v} + \nu \nabla^2 \vec{v}$$

For constant ρ, we can combine the first three terms on the right into

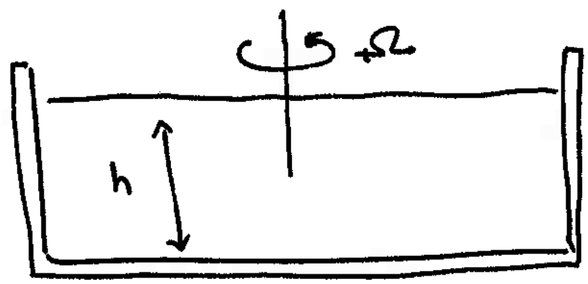
$$\varphi = \frac{1}{\rho} p - \frac{\Omega^2 x_\perp^2}{2} + gz$$

Then

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} \varphi - 2\vec{\Omega} \times \vec{v} + \nu \nabla^2 \vec{v}$$

The centrifugal force is then very easy to understand. It is just another gradient term in the Navier-Stokes equation that plays off against the pressure. It is another story with the Coriolis force. This term has some genuinely weird and counterintuitive effects.

As a first example of the effect of Coriolis forces, I will study the small oscillations of a fluid in bulk rotation. Let us ignore viscosity and take $\vec{\Omega} = \Omega\hat{z}$. Consider fluid rotating in a container that is bounded in z and very large in (x, y).



We should expand the above Navier-Stokes equation about the equilibrium state $\vec{v} = 0$, in which the cylinder of fluid rotates about $\hat{z}$ at angular velocity Ω. The equation for the first-order deviations from this state are

$$\frac{\partial v_x}{\partial t} = -\frac{\partial \varphi}{\partial x} + 2\Omega v_y$$

$$\frac{\partial v_y}{\partial t} = -\frac{\partial \varphi}{\partial y} - 2\Omega v_x$$

$$\frac{\partial v_z}{\partial t} = -\frac{\partial \varphi}{\partial z}$$

and the equation of mass conservation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

We can eliminate v_y by taking $\partial/\partial t$ of the first equation and substituting for $\partial v_y/\partial t$ from the second equation. This gives

$$\frac{\partial^2 v_x}{\partial t^2} = -\frac{\partial^2 \varphi}{\partial t \partial x} + 2\Omega\left[-\frac{\partial \varphi}{\partial y} - 2\Omega v_x\right]$$

Eliminating v_x similarly,

$$\frac{\partial^2 v_y}{\partial t^2} = -\frac{\partial^2 \varphi}{\partial t \partial y} + 2\Omega\left[+\frac{\partial \varphi}{\partial x} - 2\Omega v_y\right]$$

Now look for solutions of these equations as Fouier components in t and in $x_\perp = (x, y)$,

$$v_j = v_j(z)\, e^{-i\omega t}\, e^{i\vec{k}_\perp \cdot \vec{x}_\perp} \qquad \varphi = \varphi(z)\, e^{-i\omega t + i\vec{k}_\perp \cdot \vec{x}_\perp}$$

The equations above become

$$(-\omega^2 + 4\Omega^2)\, v_x = -\omega k_x \varphi - 2i\Omega k_y \varphi$$

$$(-\omega^2 + 4\Omega^2)\, v_y = -\omega k_y \varphi + 2i\Omega k_x \varphi$$

These combine into

$$(-\omega^2 + 4\Omega^2)\,(k_x v_x + k_y v_y) = -\omega k^2 \varphi$$

On the other hand, the mass conservation equation becomes, in Fourier components

$$i(k_x v_x + k_y v_y) + \frac{\partial v_z}{\partial z} = 0$$

Then

$$(\omega^2 - 4\Omega^2)\frac{\partial v_z}{\partial z} = -i\omega k^2 \varphi$$

The v_z equation above gives a second relation between v_z and φ, which reads in Fourier space

$$-i\omega v_z = -\frac{d\varphi}{dz}$$

Combining these two equations, we have

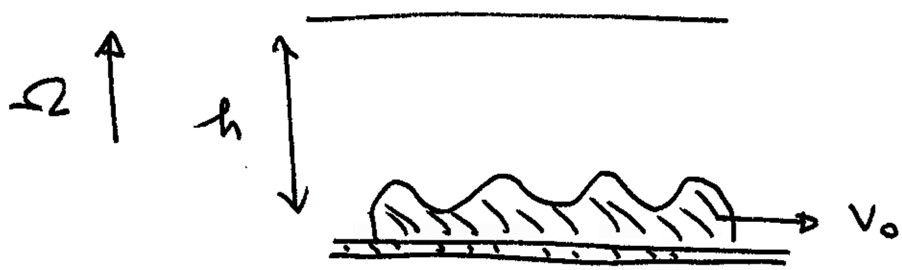
$$(\omega^2 - 4\Omega^2)\frac{d^2 v_z}{dz^2} = \omega^2 k^2 v_z$$

or, finally,

$$\frac{d^2 v_z}{dz^2} = \left[\frac{\omega^2 k^2}{(\omega^2 - 4\Omega^2)}\right] v_z$$

This equation shows a resonance at $\omega = \pm 2\Omega$ that is easy to understand as a nonlinear interaction of the perturbation with the overall relation. The consequences at low frequency are more surprising.

To probe this, imagine dragging a sinusoidal shape across the bottom of the chamber.



Effectively, this imposes a boundary condition on the fluid

$$V_z = a \sin\left[k(x - v_0 t)\right] \quad \text{at} \quad z = 0$$

The perturbation has $|\vec{k}| = k$, $\omega = kv_0$. Then the v_z responding to this boundary condition obeys

$$\frac{d^2 V_z}{dz^2} = \left(\frac{k^4 v_0^2}{(kv_0)^2 - 4\Omega^2} \right) V_z$$

If $kv_0 > 2\Omega$, v_z oscillates in (x, y) and falls off exponentially in z from the bottom of the tank to the top. This is a behavior that we have seen several times before in this course, for example, in the dependence of a gravity wave on z from the surface to the depths. However, if $kv_0 < 2\Omega$, we find a different behavior, oscillating and long-ranged in z. For $kv_0 < 2\Omega$, we have

$$\frac{d^2 V_z}{dz^2} \approx 0$$

The solution of this equation is

$$V_z = a\left(1 - \frac{z}{h}\right) \sin k(x - V_0 t)$$

From the mass conservation equation in the form

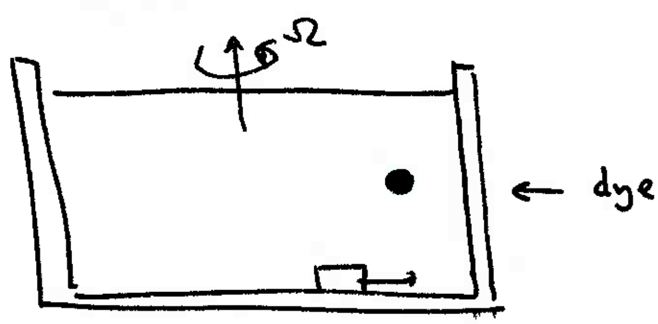
$$ik V_x + \frac{\partial V_z}{\partial z} = 0$$

we also find

$$V_x = -i \frac{a}{kh} \sin k(x - V_0 t)$$

Notice that this expression is independent of z. The column of fluid above each peak and trough of the boundary condition moves as a whole!

G. I. Taylor made an interesting demonstration of this effect: In a rotating cylinder of fluid, he put a drop of dye at a height above the bottom and slowly dragged an small solid cylinder under it.



If the dye was originally positioned just above the cylinder, it is carried along with the cylinder. If the dye was originally positioned away from the cylinder, it is pushed out of the way when the cylinder is moved under it.

The immobile column of fluid above the cylinder is called a *Taylor-Proudman column.*

There is a more formal way to understand this freezing of the dependence of the fluid velocities along the direction of the rotation. Start again from the Navier-Stokes equation. Consider steady flow (no time dependence), ignore viscosity, and neglect the nonlinear term $(\vec{v} \cdot \vec{\nabla})\vec{v}$ Then we have

$$0 = -\vec{\nabla}\varphi - 2\vec{\Omega}\times\vec{v}$$

This equation balances pressure against the Coriolis forces In components, this reads

$$\frac{\partial\varphi}{\partial x} = 2\Omega v_y$$

$$\frac{\partial\varphi}{\partial y} = -2\Omega v_x$$

Combining these equations

$$\frac{\partial^2\varphi}{\partial x\partial y} = 2\Omega\frac{\partial v_y}{\partial y} = -2\Omega\frac{\partial v_x}{\partial x}$$

Then

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

which, by mass conservation, implies,

$$\frac{\partial V_z}{\partial z} = 0$$

So, under these assumptions, any steady flow is independent of z. This result is known as *Proudman's theorem.*

The regime in which the fluid flow forms rigid columns parallel to the direction of rotation is chacterized by defining a new dimensionless variable, the *Rossby number*

$$\epsilon = \frac{V}{\Omega L}$$

where V is a characteristic velocity and L is a characteristic length. The Rossby number is an estimate of the ratio of convective to Coriolis effects.

$$\epsilon \sim \left| \frac{(\vec{V} \cdot \vec{\nabla})\, \vec{V}}{2\vec{\Omega} \times \vec{V}} \right|$$

When the Rossby number is small, convection is unimportant and the fluid flow is determined by the balance between pressure and Coriolis forces. These are the conditions under which Proudman's theorem applies. Steady flow at small Rossby number is sometimes called *geostrophic flow*, for reasons that will be discussed in the next lecture.

Now I will add back viscosity and study flows that balance viscosity against Coriolis forces. For the types of flows we have been discussing up to this point, the fluid flow is at high Reynolds number. Thus, the viscosity is only important in thin boundary layers. The boundary layers in which the Coriolis forces play an important role are called *Ekman layers.*

The effect of viscosity in a rotating fluid is characterized by another dimensionless number, the *Ekman number*

$$E = \frac{\nu}{\Omega L^2}$$

For macroscopic flows of water or air, the Ekman number is small, and so viscosity is unimportant—except in an Ekman boundary layer of thickness

$$\left(\frac{\nu}{\Omega}\right)^{1/2}$$

Ekman studied the problem of wind blowing across an ocean on a rotating earth. For simplicity here, I will study the problem of a wind blowing across a plane of water in a frame rotating about a *vertical* axis. The relation to the situation of the earth is that, in the geophysical problem, we can replce the full rotation vector with its component normal to the earth's surface. I will explain this in the next lecture.

Consider, then a fluid whose upper boundary is the plane $z = 0$, considered in a coordinate system rotating with $\vec{\Omega} = \Omega\hat{z}$. For a thin boundary layer, we can ignore pressure; I will also assume steady motion and low Reynolds number in the boundary layer. Then the Navier-Stokes equation reduces to a balance of viscous and and Coriolis forces.

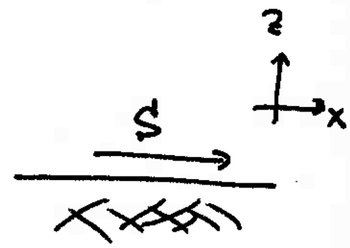
$$0 = -2\vec{\Omega} \times \vec{v} + \nu \nabla^2 \vec{v}$$

If the layer is extended in (x, y) and thin in z, we can reduce this further to

$$0 = -2\vec{\Omega} \times \vec{V} + \nu \frac{\partial^2}{\partial z^2} \vec{V}$$

For Ekman's problem, the boundary condition is that the surface is level at $z = 0$, with shear at the surface that balances the friction force applied by the wind. I will assume that this force is applied in the $\hat{x}$ direction. Then

$$\frac{\partial V_x}{\partial z} = S \qquad \frac{\partial V_y}{\partial z} = 0 \qquad V_z = 0$$



at $z = 0$. The influence of S should go to zero as $z \to -\infty$.

In components, the v_x, v_y equations are

$$0 = 2\Omega V_y + \nu \frac{d^2}{dz^2} V_x$$

$$0 = -2\Omega V_x + \nu \frac{d^2}{dz^2} V_y$$

Let

$$u = V_x + iV_y$$

then these equations can be combined into a simgle equation for this complex variable

$$\nu \frac{d^2}{dz^2} u = 2i\Omega u$$

If we set

$$k^2 = \left(\frac{\Omega}{\nu}\right)$$

we can write this equation as

$$\frac{d^2}{dz^2} u = 2ik^2 u$$

The solution of this equation that vanishes as $z \to -\infty$ is

$$u = Ae^{(1+i)kz}$$

where

$$\mathrm{Re}((1+i)k) > 0 \quad \Rightarrow \quad u \to 0 \text{ as } z \to -\infty$$

At $z = 0$, u obeys

$$\frac{\partial u}{\partial z} = S = A(1+i)k$$

so we can identify

$$A = \frac{S}{2k}(1-i)$$

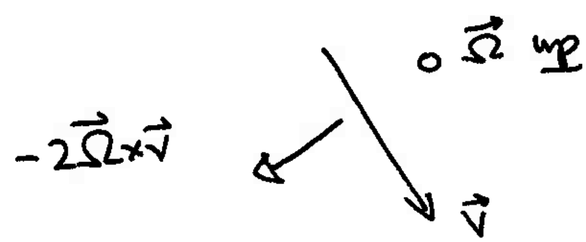
Then, finally

$$u = \frac{S}{\sqrt{2}\,k}\,\frac{1-i}{\sqrt{2}}\,e^{kz}\,e^{ikz}$$

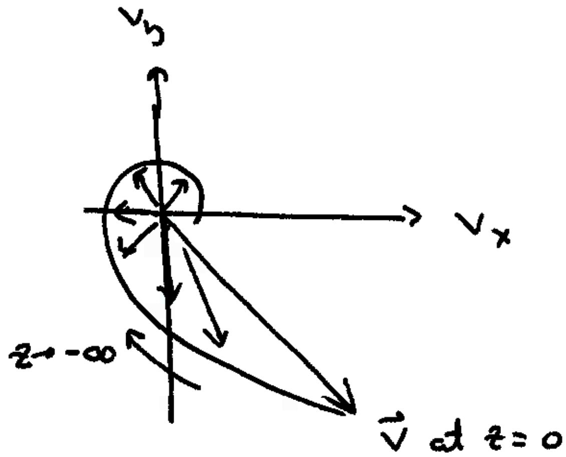
In terms of the real velocity (v_x, v_y), this is

$$(v_x, v_y) = \frac{S}{\sqrt{2}\,k}\,e^{kz}\left(\cos(kz - \pi/4),\ \sin\left(kz - \frac{\pi}{4}\right)\right)$$

It is interesting to plot this velocity profile as seen from above. At $z = 0$, the velocity that responds to the applied stress is at an angle $\phi = -\pi/4$ with respect to the direction of the stress. As we go deeper into the ocean, the vector (v_x, v_y) rotates in the negative direction The rotation in this direction as we move away from the fixed boundary value is a response to the Coriolis force



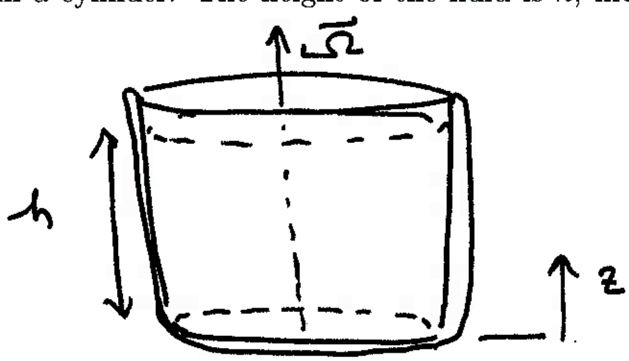
The motion of the velocity with depth then fills out the figure



This figure is called the *Ekman spiral.*

The Ekman layer plays an interesting role in a phenomenon of our common experience, the spin-down of a cup of tea that has been stirred with a spoon.

Here is an idealization of that problem: We consider a cylinder of fluid rotating uniformly in a cylinder. The height of the fluid is h, measured from the bottom at $z = 0$.



I will analyze the problem in a frame of the rotating fluid. In the intertial frame of the stationary cup, the boundary condition at the bottom of the cup is that $\vec{v} = 0$. In the rotating frame, this boundary condition is

$$\vec{v} = -\vec{\Omega} \times \vec{r}$$

at $z = 0$.

This boundary condition leads to an Ekman boundary layer at the bottom of the cup. For the fluid flow in this boundary layer, we can use the same solution as in the previous example, except that we must insist that $u(z)$ decreases in the upward, not the downward, direction. The solution for $u(z)$ is then

$$u = -\vec{\Omega}\times\vec{r}\; e^{-(1+i)kz}$$

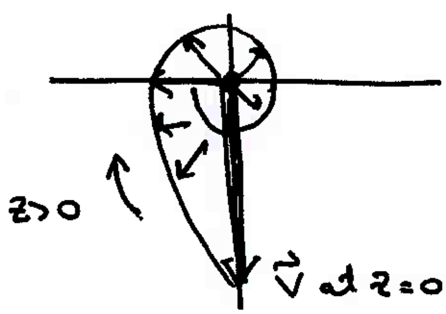
or

$$u = \Omega\,(y - ix)\; e^{-kz}\, e^{-ikz}$$

Writing $(x, y) = r(\cos\phi, \sin\phi)$, we find for the real velocities

$$(v_x, v_y) = \Omega r\, e^{-kz}\left(\sin(\phi - kz), -\cos(\phi - kz)\right)$$

At $\phi = 0$, the velocity $\vec{v}$ points in the $-\hat{y}$ direction at $z = 0$ and then, at higher z spirals in a counterclockwise direction. Here is a plot of the Ekman spiral:



Again, $\vec{v}$ turns to the right due to the Coriolis force.

In a cylinder of fluid of height h, the Ekman number is

$$E = \frac{\nu}{\Omega h^2} = \frac{1}{R}$$

For water stirred in a cup, the Reynolds number is large and the Ekman number is small. Then viscosity is effective only in an Ekman layer of thickness

$$\frac{1}{k} = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} = \left(\frac{\nu}{\Omega h^2}\right)^{\frac{1}{2}} h = E^{\frac{1}{2}} h$$

If $E \ll 1$, this is a thin layer at the bottom of the cup.

From its original orientation at $z = 0$, the velocity $\vec{v}$ in the Ekman layer turns *inward.* The radial component of velocity is

$$v_r = (\cos\phi, \sin\phi) \cdot (v_x, v_y) = -\Omega r \sin kz \, e^{-kz}$$

Then the divergence of this velocity is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = -2\Omega \sin kz \, e^{-kz}$$

If the fluid is incompressible,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Then, the above implies

$$\frac{\partial v_z}{\partial z} = +2\Omega \sin kz \, e^{-kz}$$

Since v_z is zero at $z = 0$ and the derivative of v_z starts out positive, we will have $v_z > 0$, a net outflow, at the top of the Ekman layer. To find the value of v_z at the top of the Ekman layer, we integrate the above equation

$$v_z(\text{top}) = \int_0^\infty dz \quad 2\Omega \sin kz \, e^{-kz} = \frac{\Omega}{k}$$

Thus, there is a net outflow from the Ekman layer that is approximately uniform over the bottom of the cup.

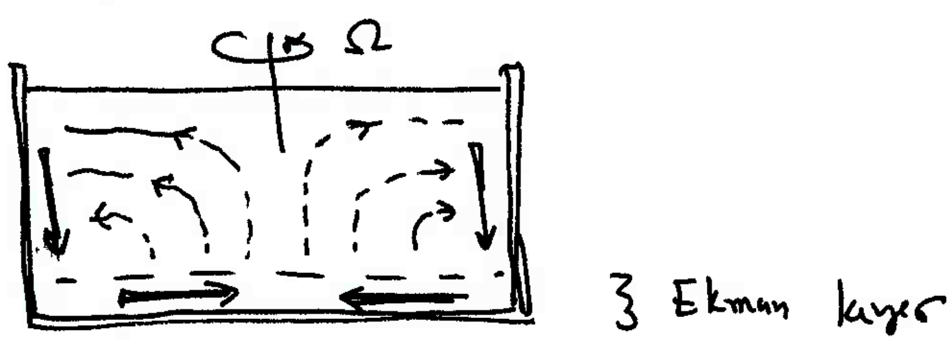
Above the Ekman layer, we have an inviscid, low Rossby number flow. Then

$$\frac{d^2 v_z}{dz^2} \cong 0 \quad \Rightarrow \quad v_z = \frac{\Omega}{k}\left(1 - \frac{z}{h}\right)$$

and so

$$-\frac{1}{r}\frac{\partial}{\partial r} r v_r = \frac{\partial v_z}{\partial z} \quad \Rightarrow \quad v_r = \frac{\Omega}{2kh} r$$

This flow leads to a slow *secondary circulation*



The velocity in this circulation is of magnitude Ω/k, which is written in terms of the Ekman number as

$$\frac{\Omega}{k} = \Omega \cdot \left(\frac{\nu}{\Omega}\right)^{1/2} = \Omega h \cdot \left(\frac{\nu}{\Omega h^2}\right)^{1/2}$$

$$= E^{1/2} (\Omega h)$$

In this sense, the secondary velocity is small, of the order of $E^{1/2}$, for small E.

Almost all of the energy of the flow is dissipated in the Ekman layer. (There is a smaller dissipation near the outside wall of the cup.) The spin-down time of the fluid can be estimated as

$$\tau \sim \frac{h^2}{\nu} \cdot \mathcal{F}^{-1}$$

where $\mathcal{F}$ is the fraction of the volume of fluid in which viscosity is active. This fraction is $E^{1/2}$; thus

$$\tau \sim \frac{h^2}{\nu} E^{-1/2}$$

You can see all of these phenomena in a cup of tea. Put in some tea leave and stir. The tea leaves visualize the flow. When the tea spins down, the tea leaves end up a small pile on the bottom of the cup in the center.