Jan. 7

Potential Flow

We can now study some explicit solutions to the fluid dynamical equations of motion. I will begin with the simplest solutions. I will assume an ideal fluid, with no internal friction, and I will add the assumptions that the fluid is *incompressible*

$$\rho = \text{const} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{v} = 0$$

and that the flow is *irrotational*

$$\vec{\omega} = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{v} = 0$$

As I discussed in the previous lecture, incompressibility is a good approximation for flows at velocities much less than the speed of sound. This is a reasonable approximation, for example, for the flow of water in common experience.

Irrotational flow means that the flow is smooth, or (*laminar*, and without eddies. Typical flows contain rotation except in very simple geometries, so this is very much a simplifying assumption. We will study the production and dynamics of vorticity later in the course.

If a flow is irrotational, $\vec{\nabla} \times \vec{v} = 0$, then the flow is described by a potential

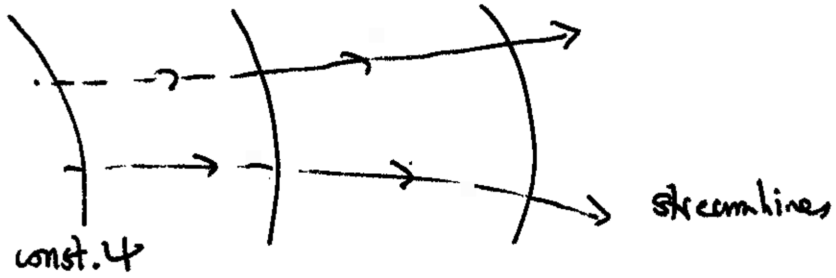
$$\vec{v} = \vec{\nabla} \psi$$

If also $\vec{\nabla} \cdot \vec{v} = 0$, then

$$\nabla^2 \psi = 0$$

These are the equations of electrostatics, so we can use our intuition from electromagnetic theory to understand these simple flows.

A potential flow can be visualized as follows: Draw the lines of constant ψ, the equipotentials. The flow follows lines perpendicular to these, *streamlines*,



The boundary condition for a potential flow is that, on a boundary, $\vec{v}$ should be parallel to the boundary:

$$\hat{n} \cdot \vec{v} = 0$$

For a fluid with friction, the friction forces will set the parallel component of $\vec{v}$ to zero just at the boundary. However, in an ideal fluid, this component is free.

The Euler equation for an ideal fluid is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \Phi$$

As in the previous lecture, we can rewrite this as

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla} v^2 - \vec{v}\times\vec{\omega} + \frac{1}{\rho}\vec{\nabla} p + \vec{\nabla}\Phi = 0$$

This was our starting point for the derivation of Bernoulli's Theorem. For an irrotational flow and constant density, we obtain an even simpler form of Bernoulli's result. The above equation becomes a pure gradient

$$\vec{\nabla}\left[\frac{\partial \psi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \Phi\right] = 0$$

so the quantity

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \Phi$$

is constant throughout the flow. For a *steady flow*, where $\vec{v}$ is independent of t, the quantity

$$B = \frac{1}{2}v^2 + \frac{p}{\rho} + \Phi$$

is constant. This gives a very simple specialization of Bernoulli's theorem.

Here are some applications of this result. First, consider a tank holding a column of water, with a small hole in the bottom.

Outside the tank, we have atmospheric pressure p_0. Assume the tank is so large or the hole so small that the velocity at the water surface at the top of the tank is negligible. Equating B at the top of the tank to its value at the spout, we find

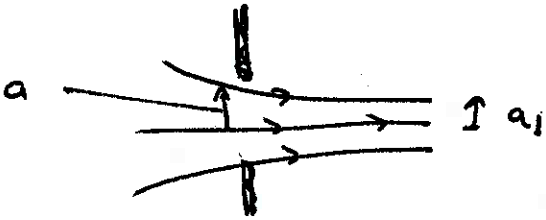
$$\frac{p_0}{\rho} + gh = \frac{1}{2}v_0^2 + \frac{p_0}{\rho}$$

Thus, the water emerges at

$$v_0 = \sqrt{2gh}$$

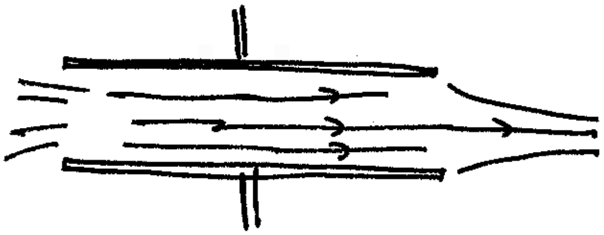
exactly as in free fall, except that the balance of pressures makes the fluid squirt out sideways. This result is *Torricelli's Theorem.*

It turns out that there is a lot to say about the form of the emerging spout. You can find a detailed discussion in Batchelor. In general, the spout narrows



A hole of radius a narrows to a stream of radius a_1. For a long, slowly converging mouthpiece, $a_1/a = 1$.

In the case that the mouthpiece is a long cylinder ('Borda's mouthpiece'), we can compute the convergence of the spout from momentum balance.



The flow through the mouthpiece is steady and all at the same height, so the momentum balance equation is

$$0 = \frac{\partial}{\partial t}(\rho v^i) = -\nabla_k T^{ik}$$

The momentum that emerges asymptotically through the spout must be equal to the momentum injected into the front of the mouthpiece. The expression for the stress tensor is

$$T^{ik} = p\,\delta^{ik} + \rho v^i v^k$$

The total momentum that comes out of the spout is given by integrating the horizontal component of the stress tensor in a region of low pressure

$$\int d^2s\, n^k T^{ik} = [p_0 + \rho v_0^2]\, \pi a_1^2$$

The momentum that goes into the spout is given by integrating the stress tensor over the front of the cylinder, where the pressure is large and the inward velocity can be ignored

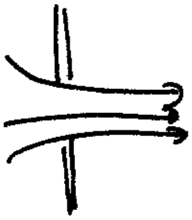
$$\int d\vec{s}\, \hat{n}^k \, T^{ik} = (p_0 + \rho g h)\, \pi a^2$$

Balancing the two contributions, we obtain $p_0 \ll \rho g h$

$$2 \rho g h \, \pi a_1^2 = \rho g h \, \pi a^2$$

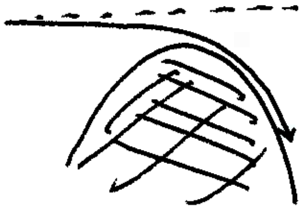
$$a_1^2 / a^2 = \tfrac{1}{2}$$

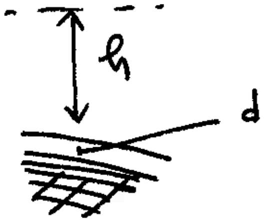
Batchelor notes that, for a simple hole in a can, observation gives



$$\left(\frac{a_1}{a}\right)^2 = 0.6$$

Another easy application of Bernoulli's principle comes in the flow of water over a dam.





I assume that the water flows in a thin sheet with a velocity approximately independent of depth at any position. We would then like to know the dependence of the thickness d of the sheet of water on horizontal position. According to Bernoulli, the velocity is given by

$$\frac{1}{2}v^2 = gh$$

The mass of water per second crossing a unit area is

$$Q = \rho v d$$

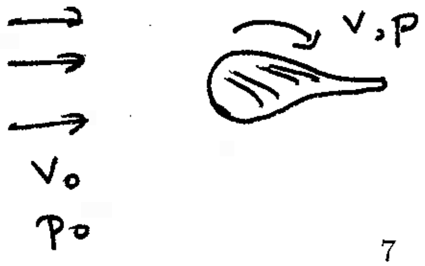
The value of Q is determined by the height of the water far to the left. However, Q is constant across the flow, and this implies

$$d = \frac{Q}{\rho v} = \frac{Q}{\rho\sqrt{2gh}} \sim \frac{1}{\sqrt{h}}$$

Finally, there is a famous application of Bernoulli's theorem to a flow of a medium over a curved surface, for example, an airplane wing. Using

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \Phi = \text{constant}$$

in steady flow, we see that an object whose geometry produces higher velocities as some points on a surface also produces lower pressures



$$p = p_0 - \frac{1}{2}\rho(v^2 - v_0^2)$$

With an airplane wing, wwe try to engineer higher velocities on the top surface to produce lift

However, the problem of producing lift is not so simple. In an irrotational flow, it turns out that the pressure difference over the whole wing integrates to zero and there is no net lift produced. We need to return to this problem later in the course when we discuss wakes and the production of vorticity.

The limit of this argument about pressures is that we can formally generate negative pressure if the velocity of the flow becomes too large. Of course, the pressure cannot really become negative. What happens instead is that the pressure becomes less than the vapor pressure of water and bubbles form; this is called *cavitation*. Batchelor has a nice picture of cavitation on a rotating propeller. It can be shown that, in irrotational flow, cavitation always begins on a surface, not in the fluid interior. Here is the argument. Applying $\vec{\nabla}$ to a formula in our derivation of Bernoulli's Theorem,

$$\nabla^2 \left[\frac{\partial \psi}{\partial t} + \tfrac{1}{2} v^2 + \frac{p}{\rho} + \Phi \right] = 0$$

Now, ψ and Φ potentials that obey $\nabla^2 \psi = 0$. Further,

$$\begin{aligned} \nabla^2 p &= \nabla^2\left(-\tfrac{1}{2}\rho v^2\right) \\ &= -\rho v^i \nabla^2 v^i - \rho (\nabla^j v^i)(\nabla^j v^i) \end{aligned}$$

Then

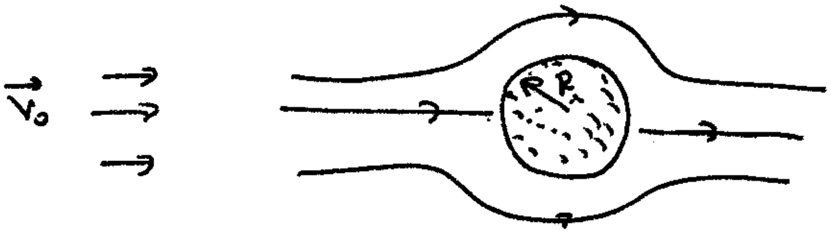
$$\nabla^2 p = -\rho (\nabla^j v^i)^2 < 0$$

so, over any small volume,

$$\int d\vec{s}\;\hat{n}\cdot\vec{\nabla}p < 0$$

Thus, the pressure cannot have a local minimum in the interior of an irrotational fluid, though it might have a local maximum.

Now I would like to work through an example in which we literally use methods from electrostatics to find the explicit form of a flow. This is the example of ideal irrotational flow around a sphere.



with $\vec{v} = \vec{v}_0$ at infinity. This is a Galilean boost of the situation in which a sphere moves through a stationary fluid at constant velocity.

We need to find ψ such that

$$\nabla^2\psi = 0 \qquad\qquad \psi \to \vec{V}_0\cdot\vec{r} \quad \text{as } \vec{r}\to\infty$$

and $\vec{v} = \vec{\nabla}\psi$ satisfies

$$\hat{r}\cdot\vec{\nabla}\psi = 0 \qquad \text{at} \qquad r = R$$

The solution will be an exterior electrostatic potential. From the symmetry of the problem, it is reasonable to try for a *dipole* potential,

$$\psi = \vec{V}_0\cdot\vec{r} + a\,\frac{\vec{V}_0\cdot\vec{r}}{r^3}$$

Then

$$\vec{\nabla}\psi = \vec{V}_0 + a\,\frac{\vec{V}_0}{r^3} - 3\,a\,\vec{V}_0\cdot\vec{r}\,\frac{\vec{r}}{r^5}$$

so that

$$\vec{v} = \vec{V}_0\left(1 + \frac{a}{r^3}\right) - 3\,\vec{V}_0\cdot\hat{r}\;\hat{r}\,\frac{a}{r^3}$$

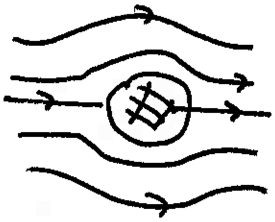
The normal component of $\vec{v}$ at the boundary is

$$\begin{aligned}\hat{r}\cdot\vec{V}_0\Big|_{r=R} &= \vec{V}_0\cdot\hat{r}\left(1 + \frac{a}{R^3}\right) - 3\,\vec{V}_0\cdot\hat{r}\,\frac{a}{R^3} \\ &= \vec{V}_0\cdot\hat{r}\left(1 - 2\,\frac{a}{R^3}\right)\end{aligned}$$

To satisfy the boundary condition, this should be zero, and that is achieved if we set

$$a = \frac{R^3}{2}$$

Then the solution of the problem is



$$\psi = \vec{V}_0 \cdot \vec{r} + \frac{1}{2}\,\frac{\vec{V}_0 \cdot \hat{r}}{r^2}\,R^3$$

$$\vec{v} = \vec{V}_0 \left(1 + \frac{1}{2}\frac{R^3}{r^3}\right) - \frac{3}{2}\,\vec{V}_0 \cdot \hat{r}\;\hat{r}\,\frac{R^3}{r^3}$$

Note that $\vec{v} = 0$ at the north and south poles of the sphere.

The pressure in the vicinity of the sphere is obtained from Bernoulli's Theorem,

$$\frac{1}{2}\rho v^2 + p = \text{constant} = \frac{1}{2}\rho v_0^2 + p_0$$

Now

$$v^2 = v_0^2 \left(1 + \frac{1}{2}\frac{R^3}{r^3}\right)^2 - 3\,(\vec{V}_0 \cdot \hat{r})^2 \left(1 + \frac{1}{2}\frac{R^3}{r^3}\right)\left(\frac{R^3}{r^3}\right)$$

$$+ \frac{9}{4}\,(\vec{V}_0 \cdot \hat{r})^2 \left(\frac{R^3}{r^3}\right)^2$$

On the surface of the sphere, we set $r = R$, $\hat{r} \cdot \vec{V}_0 = \cos\theta$. Then

$$V^2 = V_0^2 \left[\left(\tfrac{3}{2}\right)^2 - 3\,\tfrac{3}{2}\cos^2\Theta + \tfrac{9}{4}\cos^2\Theta \right]$$

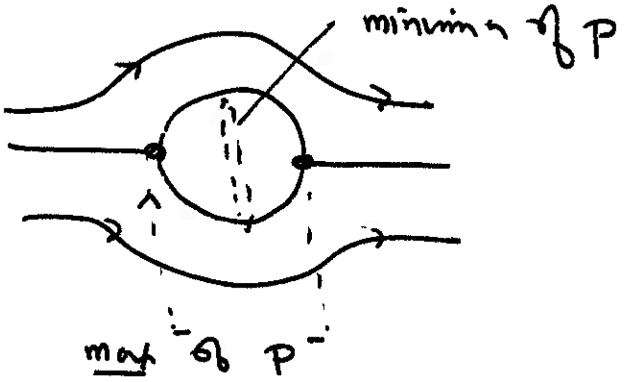
$$V^2 = \frac{9}{4} V_0^2 \left(1 - \cos^2\Theta\right)$$

and

$$p = p_0 + \tfrac{1}{2}\rho V_0^2 - \tfrac{1}{2}\rho V_0^2 \cdot \tfrac{9}{4}\left(1-\cos^2\Theta\right)$$

$$= p_0 + \frac{\rho V_0^2}{8}\left(9\cos^2\Theta - 5\right)$$

Here is a diagram of the final situation



Let's now turn our attention to flows in 2 dimensions. Among flows that have the simple incompressible, irrotational character that we are discussing here, there are many that are approximately 2-dimensional, that is, uniform in the third dimension. However, what I will emphasize now is that there is a very powerful method for finding solutions to the equations of fluid dynamics or electrodynamics in 2 dimensions that we can use to find solutions explicitly.

In 2 dimensions, we can combine the two space coordinates x and y into a complex variable

$$z = x + iy$$

An *analytic function* is one that depends on x and y only through z. (A general function of x and y would be written as a function of z and $\overline{z} = x - iy$.) I claim that, if $f(z)$ is analytic, the real and imaginary parts of $f(z)$ are harmonic functions, that is, solutions of Laplace's equation.

Here is the proof: If f is a function of $z = x + iy$ only,

$$\frac{d}{dz} f = \frac{d}{dx} f = \frac{1}{i} \frac{d}{dy} f$$

Separate f into its real and imaginary parts. Then the real and imaginary parts of this equation are

$$f = g(x,y) + i\, h(x,y)$$

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \qquad \frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y}$$

The equations are called the *Cauchy-Riemann equations*. From these it follows that

$$\frac{\partial^2}{\partial x^2} g = \frac{\partial}{\partial x} \frac{\partial h}{\partial y} = \frac{\partial}{\partial y} \frac{\partial h}{\partial x} = -\frac{\partial^2 g}{\partial y^2}$$

$$\text{so} \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) g(x,y) = 0$$

and, similarly, that $h(x, y)$ satisfies this equation also.

To solve problems in fluid mechanics, we need to construct an analytic function whose real part is the velocity potential. This can be written

$$\omega(z) = \psi(x,y) + i\, \chi(x,y)$$

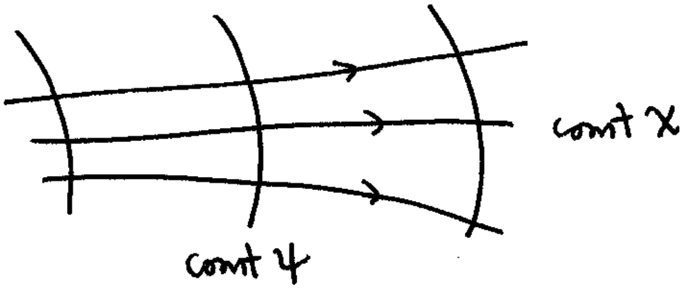
The real part $\psi(x, y)$ is the velocity potential; the imaginary part $\chi(x, y)$ is called the *stream function*.

$$\vec{v} = \vec{\nabla}\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y} \right)$$

and, using the Cauchy-Riemann equations,

$$\vec{\nabla}\chi = \left(\frac{\partial\chi}{\partial x}, \frac{\partial\chi}{\partial y} \right) = (-v_y, v_x)$$

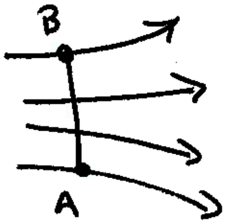
So $\vec{\nabla}\chi$ is orthogonal to $\vec{\nabla}\psi$. Thus, the lines of constant χ intersect the lines of constant ψ at right angles. These lines are then precisely the streamlines.



The boundary condition for ideal irrotational flow is satisfied if χ = constant on the boundaries.

If Q is the mass flow through a surface

$$Q = \int_A^B dl\, \hat{n}\cdot\rho\vec{v}$$



Then the flow between points A and B on different streamlines is given by

$$\frac{Q}{\rho} = \int_A^B dy\, v_x - \int_A^B dx\, v_y = \int_A^B d\vec{l}\cdot\vec{\nabla}\chi$$

or

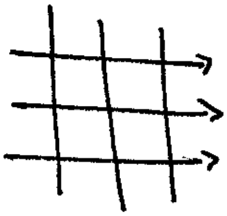
$$\frac{Q}{\rho} = \chi(B) - \chi(A)$$

A very trivial example of a function $\omega(z)$ is

$$\omega(z) = V_0 z$$

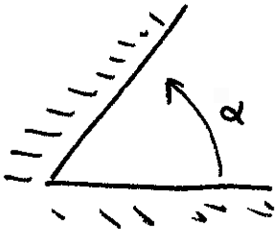
Then

$$\psi = V_0 x \qquad \chi = V_0 y \qquad v = V_0 \hat{x}$$



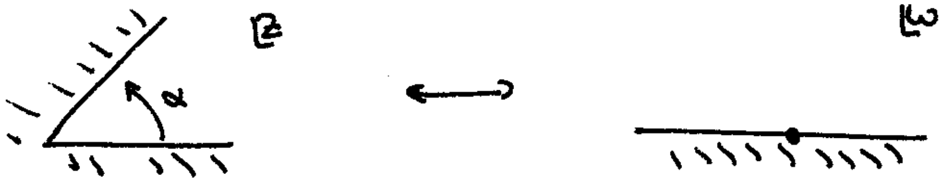
Indeed, the stream function measures the flux passing between lines of constant y.

A slightly less trivial example in the flow in a corner



We can turn this into an easy problem by making the analytic transformation, or *conformal mapping*,

$$\omega = a\, z^{\pi/\alpha}$$



For this choice, the imaginary part of ω vanishes on the boundary, so the real part of ω gives a flow that satisfies the boundary conditions. Let $\eta = \pi/\alpha$, and set $a = v_0/\eta$. We then find

$$\psi = \mathrm{Re}\, \frac{v_0}{\eta} z^{\pi/\alpha} = \frac{v_0}{\eta} r^{\eta} \cos \eta\phi$$

and, for the components of velocity,

$$v_r = \frac{\partial \psi}{\partial r} = v_0 r^{\eta-1} \cos \eta\phi$$

$$v_\phi = \frac{1}{r}\frac{\partial \psi}{\partial \phi} = - v_0 r^{\eta-1} \sin \eta\phi$$

The flow penetrates the corner as the power law $r^{\eta-1}$. Note that, for $\alpha > \pi$, this analysis is still valid and the flow is singular at the sharp corner.

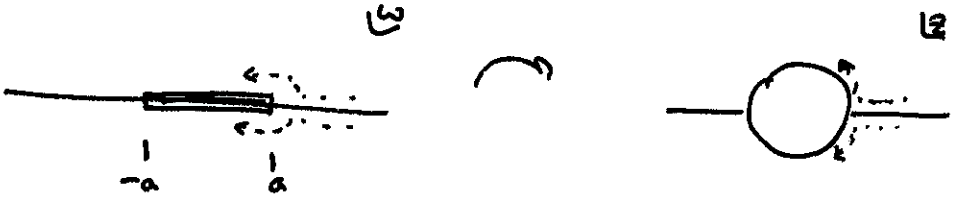
Next, consider the problem of potential flow around a cylinder. This is solved by a mapping called the *Joukowski transformation*

$$\omega = \frac{a}{2}\left(z + \frac{a^2}{z}\right) \quad \text{or} \quad z = \omega + [\omega^2 - a^2]^{\frac{1}{2}}$$

The real axis of the ω plane—the line where Im $\omega = 0$—is carried into the following shape: The piece with $\omega > a$ is carried into the real z axis with $z > a$. Similarly the real ω axis with $\omega < -a$ is carried into the real z axis with $z < -a$. The real ω axis between $-a$ and a is carried into

$$z = \omega \pm i\left[a^2-\omega^2\right]^{\frac{1}{2}}$$

which is a circle of radius a. Then



Finally, $\omega \sim z$, giving a constant horizontal velocity, for $z \to \infty$. Then $\omega(z)$ gives the solution that we are looking for. From

$$\omega = \frac{a}{2}\left(z + \frac{a^2}{z}\right)$$

the real part is

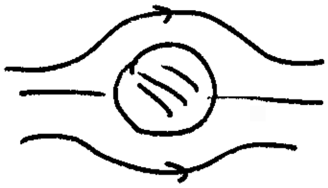
$$\psi = \mathrm{Re}\,\omega = \frac{a}{2}\left[x + \frac{a^2 x}{x^2+y^2}\right]$$

This problem is linear, so we can rescale ψ to a fixed asymptotic velocity,

$$\psi = V_0\left(x + \frac{a^2 x}{x^2+y^2}\right) \simeq \vec{V}_0\vec{r} + \frac{a^2}{r^2}\vec{V}_0\cdot\vec{r}$$

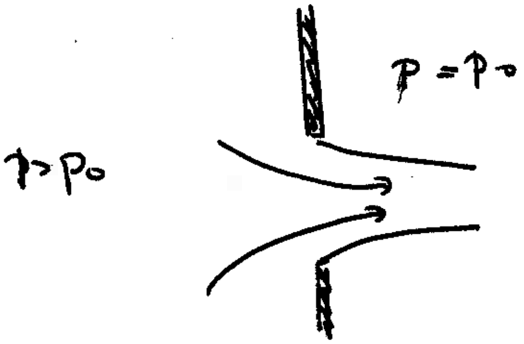
This is a dipole potential in 2 dimensions. The velocity distribution is similar to that for a sphere,

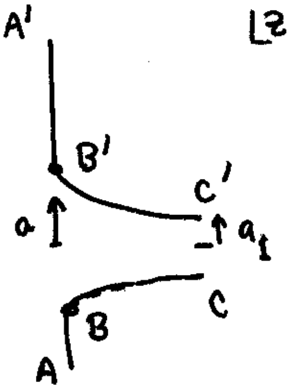
$$\vec{V} = \vec{\nabla}\psi = \vec{V}_0 \left(1+\frac{a^2}{r^2}\right) - 2\,\frac{a^2}{r^2}\,\vec{V}_0\cdot\hat{r}\;\hat{r}$$



As a final application of conformal mapping, I would like to take you through a quite sophisticated example presented in Landau and Lifshitz to solve the problem of the shape of a spout in 2 dimensions. This is a real magic trick.

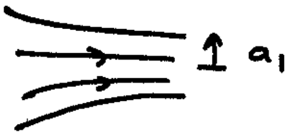
Here is the problem that we would like to solve





Accourding to Bernoulli's Theorem, $B = \frac{1}{2}v^2 + p/\rho$ is constant. The curves BC and $B'C'$ are at atmospheric pressure p_0, so $|\vec{v}|$ is constant along these curves. The curves ABC and $A'B'C'$ are streamlines, so χ is also constant along these curves. The mass flow out the spout is

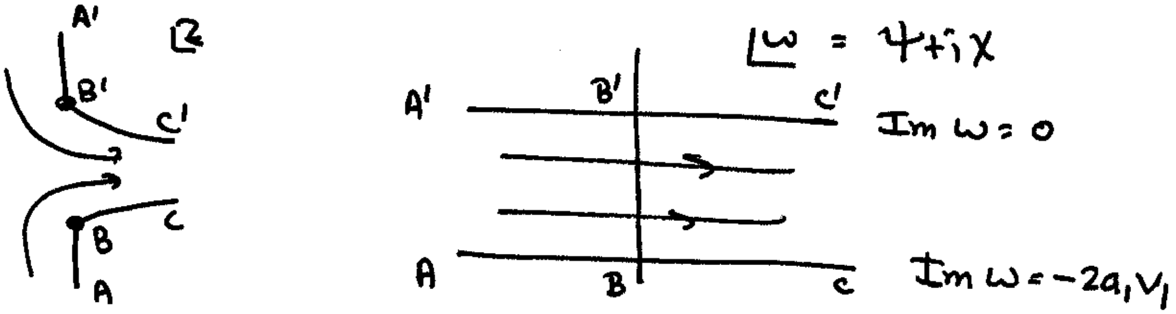
$$Q = \rho V_1 \cdot 2a_1$$



where v_1 is the velocity at the spout far away from the hole, which must equal the velocity on BC and $B'C'$. Then

$$\chi(A'B'C') - \chi(ABC) = \frac{Q}{\rho} = 2a_1 v_1$$

Now, $\chi = \operatorname{Im}\omega$, so the mapping from the z plane to ω is the qualitative form



We do not know $\omega(z)$, but we do know something about $d\omega/dz$. From the equations at the start of this discussion

$$\frac{d\omega}{dz} = \frac{\partial\psi}{\partial x} + i\,\frac{\partial\chi}{\partial x} = v_x - i v_y \equiv v e^{-i\phi}$$

where $v = |\vec{v}|$ and ϕ is the orientation of $\vec{v}$. Call this complex number v, so that

$$\frac{d\omega}{dz} = v$$

Then v has the properties

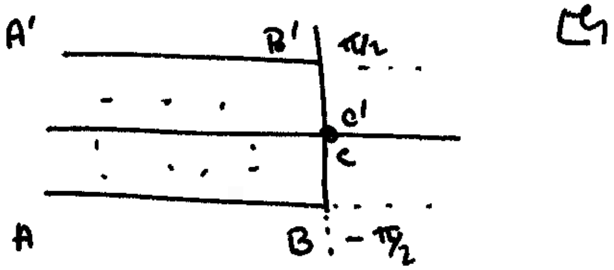
$$|v| = v_1 \quad \text{on} \quad BC\,,\ B'C'$$

$$\phi = -\frac{\pi}{2} \quad \text{on} \quad A'B' \qquad \phi = +\frac{\pi}{2} \quad \text{on} \quad AB$$

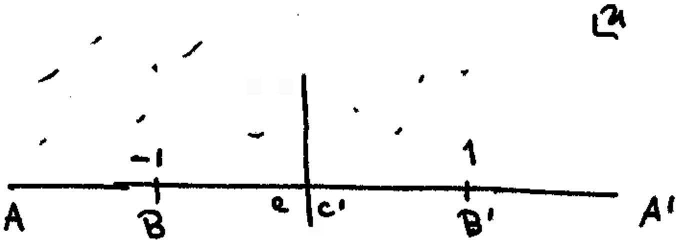
Now let

$$\zeta = \log \frac{d\omega}{dz} - \log v_1$$

The image of ω in the ζ plane occupies the following region:



We can map both of these regions from the half-plane



by

$$\omega = \frac{2a_1 v_1}{\pi} \log \frac{1}{u}$$

$$\zeta = i \sin^{-1} u \quad \text{or} \quad \log \frac{d\omega}{dz} = \log v_1 + i \sin^{-1} u$$

In principle, the problem is now solved. We have ω and $d\omega/dz$ as parametric functions of u, and now we just need to eliminate u.

Let's try to make this more explicit along the line BC.

$$\frac{d\omega}{dz} = V_1 e^{-i\phi} \qquad \phi = +\tfrac{\pi}{2} \text{ at } B \quad = 0 \text{ at } C$$

$$dz = \frac{1}{V_1} e^{i\phi} \, d\omega$$

also

$$\eta = \log \frac{d\omega}{dz} - \log V_1 = -i\phi = i \sin^{-1} u$$

so that

$$u = -\sin\phi$$

The equation for ω in terms of u gives

$$d\omega = -\frac{2a_1V_1}{\pi} \frac{1}{u} \, du = -\frac{2a_1V_1}{\pi} \frac{1}{\sin\phi} \cos\phi \, d\phi$$

Then

$$dz = -\frac{2a_1}{\pi} e^{i\phi} \frac{\cos\phi}{\sin\phi} \, d\phi$$

Along BC, $\operatorname{Im} z$ increases from $(-a)$ to $(-a_1)$,

$$a - a_1 = \int_B^C d\,\mathrm{Im}\,z = \int_{\pi/2}^{0} \left(-\frac{2a_1}{\pi}\cos\phi\, d\phi\right) = \frac{2a_1}{\pi}$$

Thus

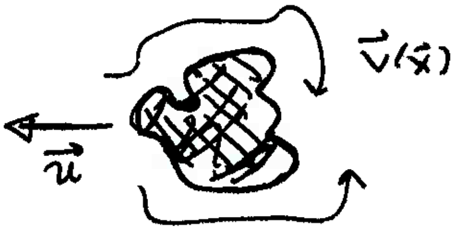
$$a = a_1\left(1 + \frac{2}{\pi}\right)$$

or, finally

$$\frac{a_1}{a} = \frac{\pi}{\pi+2} \mathrel{\hat{=}} 0.61$$

This value can be compared to those quoted earlier in this lecture.

Now I would like to develop a general theory of a solid object moving in an incompressible, irrotational fluid.



Let the velocity of the object be $\vec{u}$, and the velocity of the fluid moving around the object be $\vec{v}(\vec{x})$.

Because of the assumption of *incompressibility*, the response of the fluid to a displacement of the body is instantaneous. Recoemmber that, by assuming incompressibility, we are effectively assuming the the velocities $\vec{u}$, $\vec{v}$ are much less than the speed of sound in the fluid.

For a sphere of radius R, we can find the solution by making a Galilean boost of the formulae we found earlier,

$$\psi = -\frac{1}{2}\frac{\vec{u}\cdot\hat{r}}{r^2}R^3$$

$$\vec{v} = \frac{R^3}{2r^3}\left(3\vec{u}\cdot\hat{r}\,\hat{r} - \vec{u}\right)$$

This is a pure electrostatic dipole. That suggests a general method of analysis.

Outside the object, the flow is given by a velocity potential satisfying

$$\nabla^2\psi = 0$$

So, far away from the object, we can always represent ψ by a multipole expansion

$$\psi = -\frac{Q}{r} - \frac{\vec{A}\cdot\vec{r}}{r^3} - \frac{Q_{ij}\vec{r}_i\vec{r}_j}{r^5} + \cdots$$

If $Q \neq 0$, the asymptotic behavior is dominated by the monopole term. But this would give

$$\vec{\nabla}\psi \cong +Q\frac{\hat{r}}{r^2}$$

which is a uniform flow out of a large sphere. If the fluid is incompressible and the object is rigid, there cannot be a net outflow from the neighborhood of the object. Thus, $Q = 0$.

Then

$$\psi = -\frac{\vec{A}\cdot\hat{r}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

$$\vec{v} = \frac{3\vec{A}\cdot\hat{r}\;\hat{r} - \vec{A}}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)$$

For an object whose shape is complicated, we have to solve a nontrivial electrodynamics problem to compute $\vec{A}$.

Using the multipole expansion, we can compute the energy and momentum of the fluid. To begin, consider the energy in a large sphere of radius R. Eventually, I will take the limit $R \to \infty$. Let V be the volume of the large sphere and V_0 be the volume of the solid object. Then

$$E = \int_{V-V_0} d^3x \quad \frac{1}{2}\rho v^2$$

Now

$$v^2 = u^2 + (v^2 - u^2) = u^2 + (\vec{v}+\vec{u})\cdot(\vec{v}-\vec{u})$$

$$= u^2 + \vec{\nabla}(\psi + \vec{u}\cdot\vec{r})\cdot(\vec{v}-\vec{u})$$

or

$$v^2 = u^2 + \vec{\nabla}\left[(\psi + \vec{u}\cdot\vec{r})(\vec{v}-\vec{u})\right]$$

$$\text{since} \quad \vec{\nabla}\cdot\vec{v} = 0$$

Integrating the terms here separately,

$$\int_{V-V_0} d^3x \quad \tfrac{1}{2}\rho u^2 = (V-V_0)\,\tfrac{1}{2}\rho u^2$$

$$\int_{V-V_0} d^3x \quad \tfrac{1}{2}\rho\vec{\nabla}\left[(\psi+\vec{u}\cdot\vec{r})(\vec{v}-\vec{u})\right]$$

$$= \int_{\partial V} d^2s \; \tfrac{1}{2}\rho\; \hat{n}\cdot\; (\psi+\vec{u}\cdot\vec{r})(\vec{v}-\vec{u})$$

$$- \int_{\partial V_0} d^2s \; \tfrac{1}{2}\rho\,(\psi+\vec{u}\cdot\vec{r})\; \hat{n}\cdot(\vec{v}-\vec{u})$$

The boundary condition on the boundary of the object is

$$\vec{n}\cdot\vec{v} = \hat{n}\cdot\vec{u}$$

so the second term here vanishes. The first term is an integral over a sphere of radius R, with $\hat{n}=\hat{r}$. The area of the sphere is $4\pi R^2$, so we can ignore any term that falls faster than $1/r^2$ as $r\to\infty$. Using this insight, we can represent the integrand of the first term as

$$\hat{r}\cdot(\psi+\vec{u}\cdot\vec{r})(\vec{v}-\vec{u})$$

$$= \left[-\frac{A\hat{r}}{r^2}+O\left(\frac{1}{r^3}\right)\right]\left[-\hat{r}\vec{u}+O\left(\frac{1}{r^3}\right)\right.$$

$$\left.+(\vec{u}\cdot\vec{r})\left(\hat{r}\frac{(3\vec{A}\cdot\hat{r}\hat{r}-\vec{A})}{r^3}+O\left(\frac{1}{r^4}\right)-\hat{r}\cdot\vec{u}\right)\right.$$

$$= \frac{\vec{A}\cdot\hat{r}\;\vec{u}\cdot\hat{r}}{r^2}+\frac{2\vec{A}\cdot\hat{r}\;\vec{u}\cdot\hat{r}}{r^2}-\vec{u}\cdot\vec{r}\;\hat{r}\cdot\vec{u}+O\left(\frac{1}{r^3}\right)$$

To integrate this over the large sphere, use

$$\int d^2s \; \vec{A}\cdot\hat{r} \; \vec{B}\cdot\hat{r} = 4\pi R^2 \cdot \tfrac{1}{3}\vec{A}\cdot\vec{B}$$

$$\int_{\partial V} d^2s \; \tfrac{1}{2}\rho\hat{r}\,(\psi+\vec{u}\cdot\vec{r})(\vec{v}-\vec{u})$$

$$= \tfrac{1}{2}\rho \, 4\pi R^2 \left[\tfrac{1}{3}\frac{\vec{A}\cdot\vec{u}}{R^2} + \tfrac{2}{3}\frac{\vec{A}\cdot\vec{u}}{R^2} - \tfrac{1}{3}u^2 R \right] + \dots$$

$$= \tfrac{1}{2}\rho \, 4\pi \, \vec{A}\cdot\vec{u} - \tfrac{1}{2}\rho \cdot \tfrac{4\pi}{3}R^3 u^2$$

Adding the pieces

$$E = \tfrac{1}{2}\rho \left[4\pi \vec{A}\cdot\vec{u} - V_0 u^2 \right]$$

It is a linear problem to solve for $\vec{A}$, so $\vec{A}$ is linearly related to $\vec{u}$,

$$A_i = a_{ij} u_j$$

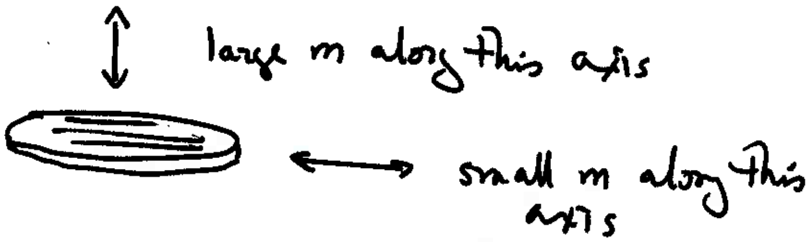
Then we can define

$$m_{ij} = \rho \left(4\pi a_{ij} - V_0 \delta_{ij} \right)$$

and rewrite this equation as

$$E = \tfrac{1}{2} m_{ij} u_i u_j$$

The tensor m_{ij} is the *effective mass* that the object acquires by virtue of the inertia of the fluid that it must push out of the way. The tensor structure of m_{ij} reflects the shape of the object. A disk has



For a sphere

$$\vec{A} = \frac{R^3}{2}\vec{u}$$

so

$$m_{ij} = \left(4\pi\frac{R^3}{2}\delta_{ij} - \frac{4\pi}{3}R^3\delta_{ij}\right)\rho$$

or

$$m_{ij} = \frac{2\pi}{3}R^3\rho\quad\delta_{ij}\qquad\text{isotropic}$$

To compute the momentum of the fluid, note that changes in momentum and energy are related by

$$dE = d\vec{u}\cdot\vec{P}$$

so

$$P_i = m_{ij} u_j$$

If an external forces is applied to the object, Newton's law tells us that

$$\frac{d}{dt} P_i = \frac{dP_i}{dt}(\text{object}) + \frac{dP_i}{dt}(\text{fluid}) = \left(\begin{array}{c}\text{external}\\ \text{force}\end{array}\right)$$

or

$$\frac{d}{dt} P_i = (M \delta_{ij} + m_{ij})\, \dot{u}_j = f_i$$

where M is the mass of the object. Then the equation of motion of an object immersed in a fluid is

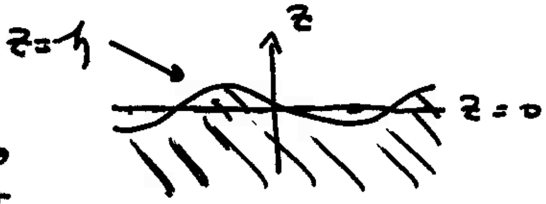
$$(M\delta_{ij} + m_{ij}) \frac{du_j}{dt} = f_i$$

Notice that there is no *dissipation* in this equation. This is a result of our assumptions of ideal fluid behavior and instantaneous response. In a compressible fluid, even if it is ideal, a moving object will radiate sound waves.

As a final topic in this discussion of potential flow, I would like to discuss *waves* on the surface of an ideal fluid. An incompressible fluid does not allow waves to propagate in its bulk, but it can carry waves on its surface. I will consider waves of small amplitude, so that we can neglect the nonlinear term $(\vec{v} \cdot \vec{\nabla})\vec{v}$ in the Euler equation.

In equilibrium, the surface of the fluid will be horizontal, and we can place it at $z = 0$. In general, let $\zeta(x, y, t)$ be the z-coordinate of the free surface. The pressure is p_0 at $z = \zeta$. For simplicity, I will consider waves that are moving in the $\hat{x}$ direction.

The Euler equation



$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p + \rho \vec{g}$$

implies

$$\vec{\nabla}\left(\rho \frac{\partial \psi}{\partial t} + p + \rho g z\right) = 0$$

so on the surface

$$\rho \frac{\partial \psi}{\partial t} + p_0 + \rho g \zeta(x,t) = \mathrm{const}$$

A wavelike solution to this equation is

$$\psi = -\frac{p_0}{\rho} t + a \cos(\omega t - kx) f(z)$$

with

$$\zeta = b \sin(\omega t - kx)$$

The term proportional to t has no effect on the wave. For potential flow, $\nabla^2\psi = 0$. Then

$$\left[-a k^2 f(z) + a f''(z) \right] \cos(\omega t - kx) = 0$$

Thus

$$f(z) = A e^{kz} + B e^{-kz}$$

The boundary condition is $\vec{v} \to 0$ as $z \to -\infty$, so

$$\psi = -\frac{p_0}{\rho} t + a \cos(\omega t - kx) \; e^{kz}$$

The fluid velocity at the surface is

$$v_z = \frac{\partial \psi}{\partial z} = \frac{\partial \zeta}{\partial t}$$

which implies

$$ak \cos(\omega t - kx) = b\omega \cos(\omega t - kx)$$

The Euler equation at the surface gives for the oscillating part

$$\frac{\partial \psi}{\partial t} + g h = 0$$

which implies

$$-a\omega \sin(\omega t - kx) + bg \sin(\omega t - kx) = 0$$

Then we find the relations for k and ω

$$ak = b\omega \qquad a\omega = bg$$

We can eliminate a/b to find the dispersion relation

$$\omega^2 = gk$$

Here is a picture of the fluid motion in the wave: