Jan. 21

Viscous Flow (continued)

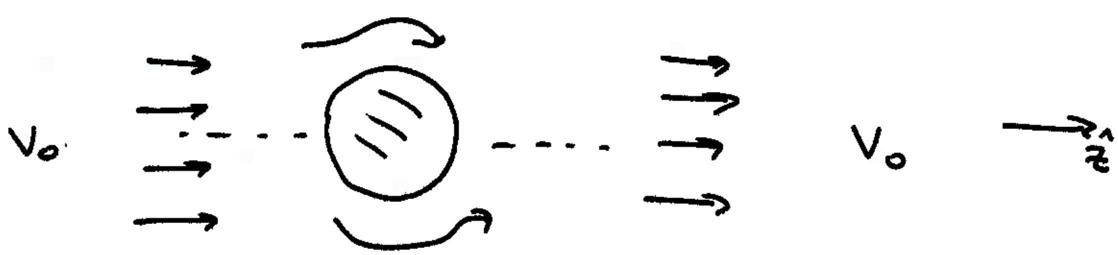
To finish our study of simple solutions of the Navier-Stokes equation, I will present on of the classic examples in fluid mechanics, the flow of a fluid around a sphere in the regime of high viscosity. Even in Physics 1, you often see the *Stokes formula* for the drag force on a sphere

$$\vec{F} = -6\pi\eta R\,\vec{V}$$

I will now derive this result.

The derivation will be given for what we might call the *Stokes regime* where we can ignore the nonlinear term in the Navier-Stokes equation, $(\vec{v}\cdot\vec{\nabla})\vec{v}$, relative to the viscous term $\nu\nabla^2\vec{v}$. In the previous examples, the special geometries that we used allowed us to derive this, but here it is an assumption or a restriction. I will say more later in the lecture about when this restriction applies.

We can set up the problem as we did the problem of idea fluid motion around a sphere. We put the sphere, of radius R, at rest, and consider a fluid flowing around it with velocity $\vec{V}_0$ at infinity.



This is a Galilean boost of the situation where the sphere moves through a stationary fluid. We will look for a steady flow. The equations to be satisfied are then

$$0 = -\frac{\vec{\nabla}p}{\rho} + \nu\nabla^2\vec{V} \qquad\qquad \vec{\nabla}\cdot\vec{V} = 0$$

The solution is called *Stokes flow.*

Taking $\vec{\nabla}$ dotted into the first equation, we find

$$\nabla^2 p = 0$$

Thus, in Stokes flow, the pressure is a harmonic function. We can solve the problem by making an appropriate ansatz for p and then trying to build up $\vec{v}$ from it. The vorticity $\vec{\omega}$ satisfies

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v} = -\nabla^2 \vec{v}$$

so it follows that

$$\frac{\vec{\nabla} p}{\rho} = -\nu \vec{\nabla} \times \vec{\omega}$$

A reasonable strategy is to go from p to $\vec{\omega}$ and then to $\vec{v}$.

Since p is harmonic outside the sphere, $r > R$, we can represent it by a multipole expansion,

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{Q}{r} - \frac{\vec{A} \cdot \vec{r}}{r^3} - \ldots$$

If $Q \neq 0$, the asymptotic behavior of p is

$$\vec{\nabla}\frac{p}{\rho} = \frac{Q\hat{r}}{r^2} + \ldots$$

which is wrong, because p should decrease from $z \to -\infty$ to $z \to \infty$. Try, then

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{\vec{A}\cdot\hat{r}}{r^2}$$

The constant $\vec{A}$ should presumably be parallel to $\vec{V}_0$. Write

$$\vec{A} = \nu\vec{B} \quad \text{with} \quad \vec{B} \parallel \vec{V}_0$$

Then

$$\frac{\vec{\nabla}p}{\rho} = -\frac{\nu}{r^3}\left(\vec{B} - 3\,\vec{B}\cdot\hat{r}\,\hat{r}\right) = -\nu\vec{\nabla}\times\vec{\omega}$$

We now have $\vec{\nabla}\times\vec{\omega}$, so we can find $\vec{\omega}$ with the requirement that $\vec{\omega} \to 0$ as $r \to \infty$. I propose

$$\vec{\omega} = \frac{\vec{r}\times\vec{B}}{r^3}$$

Let's check this

$$\vec{\nabla}\times\vec{\omega} = \frac{1}{r^3}\left[\vec{\nabla}\times(\vec{r}\times\vec{B}) - 3\,\hat{r}\times(\hat{r}\times\vec{B})\right]$$

$$= \frac{1}{r^3}\left[\vec{B}\cdot\vec{\nabla}\,\vec{r} - (\vec{\nabla}\cdot\vec{r})\vec{B} - 3\,\hat{r}(\hat{r}\cdot\vec{B}) + 3\vec{B}(\hat{r})^2\right]$$

$$= \frac{1}{r^3}\left[\vec{B} - 3\vec{B} - 3\,\hat{r}(\hat{r}\cdot\vec{B}) + 3\vec{B}\right]$$

$$= \frac{1}{r^3}\left[\vec{B} - 3\,\hat{r}\,\hat{r}\cdot\vec{B}\right]$$

Now we can find $\vec{v}$ by solving the equation $\vec{\nabla}\times\vec{v} = \vec{\omega}$.

$$\vec{\nabla}\times\vec{v} = \frac{\hat{r}\times\vec{B}}{r^2}$$

A solution to this equation is

$$\vec{v} = -\frac{\vec{B}}{r}$$

but this does not satisfy the boundary condition that $\vec{v} = 0$ at $f = R$. Note that, this time, all components of $\vec{v}$ must vanish on the boundary of the sphere. To fix the boundary condition, we have the freedom to add to $\vec{v}$ a gradient, for which the curl is automatically zero,

$$\vec{v} = -\frac{\vec{B}}{r} + \vec{\nabla}\lambda(r)$$

The function λ should depend linearly on $\vec{B}$; an appropriate ansatz is

$$\lambda = \vec{B} \cdot \vec{r} \; g(r)$$

Then

$$\vec{v} = -\frac{\vec{B}}{r} + \vec{B} \, g(r) + \vec{B} \cdot \vec{r} \; \vec{\nabla} g$$

The function $g(r)$ is constrained by $\vec{\nabla} \cdot \vec{v} = 0$.

$$0 = \vec{\nabla} \cdot \vec{v} = \frac{\hat{r} \cdot \vec{B}}{r^2} + 2 \vec{B} \cdot \hat{r} \frac{dg}{dr} + \vec{B} \cdot \vec{r} \; \nabla^2 g$$

$$= \vec{r} \cdot \vec{B} \left[\frac{1}{r^3} + \frac{2}{r} \frac{dg}{dr} + \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} g \right]$$

This implies

$$\frac{d^2 g}{dr^2} + \frac{4}{r} \frac{dg}{dr} + \frac{1}{r^3} = 0$$

The solution to this equation is

$$g(r) = \frac{1}{2r} + a + \frac{b}{r^3}$$

for constants a, b. Then

$$\vec{v} = -\frac{\vec{B}}{r} + \vec{B}\left(a + \frac{1}{2r} + \frac{b}{r^3}\right) + \vec{B}\cdot\vec{r}\,\hat{r}\left(-\frac{1}{2r^2} - \frac{3b}{r^4}\right)$$

The parameters a and b give us enough freedom to set $\vec{v} = 0$ at $r = R$. First, from the coefficient of $\vec{B} \cdot \vec{r}\,\hat{r}$,

$$-\frac{1}{2r^2} - \frac{3b}{r^4}\bigg|_{r=R} = 0 \quad \Rightarrow \quad b = -\frac{1}{6}R^2$$

Then, from the coefficient of $\vec{B}$,

$$\left(-\frac{1}{r} + a + \frac{1}{2r} - \frac{1}{6}\frac{R^2}{r^3}\right)\bigg|_{r=R} = 0 \quad \Rightarrow \quad a = \frac{2}{3R}$$

The asymptotic behavior $\vec{v} \to \vec{V}_0$ as $r \to infty$ fixes $\vec{B}$.

$$\vec{v} \to \vec{B}\cdot a = \vec{V}_0 \quad \text{or} \quad \vec{B} = \frac{3}{2}R\vec{V}_0$$

Thus, finally, we find the solution

$$\vec{v} = \vec{V}_0\left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right] - \frac{3}{4}\vec{V}_0\cdot\hat{r}\,\hat{r}\,\frac{R}{r}\left(1 - \frac{R^2}{r^2}\right)$$

The equation above for the pressure becomes

$$p = p_\infty - \frac{3}{2}\eta \frac{\vec{V}_0\cdot\hat{r}}{r^2} R$$

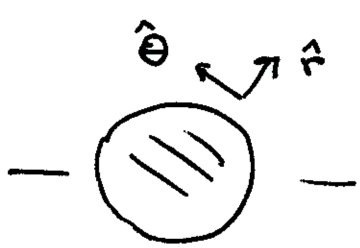
From these results, we can compute the forces that the fluid exerts on the sphere. The forces come from two sources, the pressure and the viscous stresses. The force due to the pressure is in the direction normal to the sphere. Integrating over the surface,

$$F_z^{(p)} = \hat{z}\cdot\int d^2s\,(-\hat{r}\,p) = 2\pi R^2\int_{-1}^{1} d\cos\theta\,(-\cos\theta)\left(-\frac{3}{2}\eta\frac{V_0 R}{R^2}\cos\theta\right)$$

The force due to the viscosity is given by

$$F_z^{(\eta)} = \hat{z}^i \int d^2s\; \hat{n}^j\,(2\eta\sigma^{ij})$$

On the surface of the sphere, $\vec{v} = 0$, so this has no normal component. However, there is a shear, $\partial\vec{v}/\partial r \neq 0$, which provides a force in the (tangential) direction of $\vec{v}$. Using the geometry



this force is given by

$$F_z^{(3)} = \int d^2s \quad \hat{z}\cdot\hat{\theta} \quad \eta \, \frac{\partial v_\theta}{\partial r}$$

Then, since

$$\hat{z}\cdot\hat{\theta} = -\sin\theta \qquad\qquad v_\theta = -V_0 \sin\theta \left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right]$$

$$\left.\frac{\partial v_\theta}{\partial r}\right|_{r=R} = -V_0 \sin\theta \left(\frac{3}{4} + \frac{3}{4}\right)\frac{1}{R}$$

the force in the $\hat{z}$ direction is given by the integral

$$F_z^{(m)} = 2\pi R^2 \int_{-1}^{1} d\cos\theta \;(-\sin\theta)\; \eta \;\left(-V_0 \frac{3}{2}\sin\theta\right)\frac{1}{R}$$

The total force in the $\hat{z}$ direction is then

$$F_z = 2\pi R \cdot \frac{3}{2} V_0 \eta \int_{-1}^{1} d\cos\theta \;(\cos^2\theta + \sin^2\theta)$$

so that indeed

$$\vec{F} = \hat{z}\cdot 6\pi\eta R\, V_0$$

This force, in the positive $\hat{z}$ direction, would represent a *drag* on the sphere in the frame where the sphere is moving through the fluid.

I should now return to a general issue that we have left hanging. We defined the Stokes regime as the regime of fluid flow in which

$$\left| (\vec{v} \cdot \vec{\nabla}) \vec{v} \right| \ll \left| \nu \nabla^2 \vec{v} \right|$$

What is the criterion for being in this regime? We can analyze this question in the following way: For a particular flow that we might be trying to study, let L be the typical length scale and V the typical scale of velocities. Then this criterion is estimated by

$$\frac{V^2}{L} \ll \nu \frac{V}{L^2}$$

or

$$R = \frac{VL}{\nu} \ll 1$$

The dimensionless quantity R that appears here is called the *Reynolds number.* Flows with low Reynolds number are dominated by viscosity; flows with high Reynolds number are dominated by convection.

There is a more general principle at work here. Let's write out the Navier-Stokes equation once again:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}$$

Now replace the coordinates and velocities with dimensionless variables with the characteristic lengths and speeds scaled out,

$$x = L\,x' \qquad t = t' \cdot \frac{V}{L} \qquad \vec{v} = V\,v'(x',t')$$

Along with this, we can rescale the pressure. Remember that, when we solve for a flow, we do not specify the pressure independently but, rather, we solve for it using the Navier-Stokes equation combined with the relation of incompressibility. So we are free to define a rescaled pressure in any convenient way, for example,

$$p - p_0 = \rho V^2 \cdot p'(x',t')$$

Then notice that a factor of V^2/L scales out of the Navier-Stokes equation uniformly, leaving behind the equation

$$\frac{\partial v'}{\partial t'} + (\vec{v}' \cdot \vec{\nabla}')\,\vec{v}' = -\vec{\nabla}' p' + \frac{1}{R}\,\nabla'^2 \vec{v}'$$

All dependence on dimensionful quantities has been removed, except for the dependence on scales in the coefficient of the viscosity term. Here we again find the Reynolds number,

$$R = \frac{VL}{\nu} = \frac{\rho V L}{\eta}$$

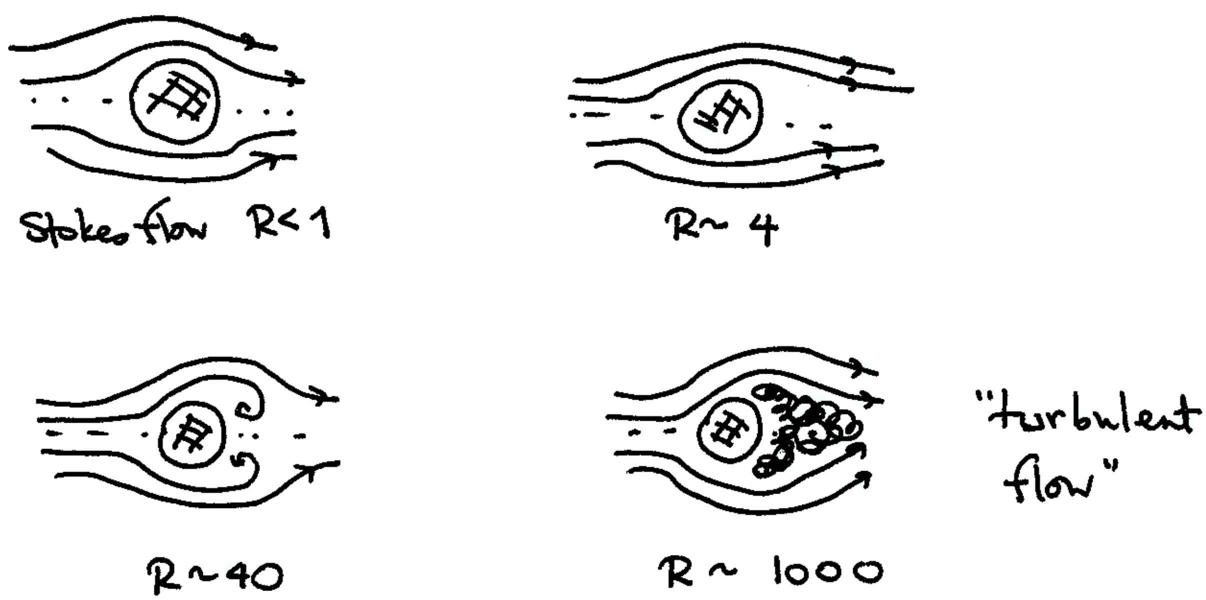
We have now shown that flows at the same value of R, even with every different values of L or ν, satisfy the same equation and thus must be identical up to rescaling. Again, small R implies that the viscosity dominates, while at large R the the convection term is more important.

In the early history of fluid dynamics, R was used to guide the creation of scale models. For example, the drag force D on a boat has the units of g cm/sec^2 $\sim \rho V^2 L^2$. So

$$\frac{D}{\rho V^2 L^2} = \text{function of } R \text{ only}$$

To find the drag force for a complicated design, one need only construct a scale model and flow with the same value of R and then use this model as an analog computer for the drag force.

If we fix the shape of an object and then set up a flow around it at higher and higher velocities, we move from small to large R. You might then expect that the flow would smoothly transition from a viscous Stokes flow at low R to an ideal potential flow at large R. This not what happens. Batchelor has a nice set of photographs of fluid flow around a cylinder at a set of increasing values of R. As R increases, the flow stays laminar up to a point but then becomes increasingly complex in the region behind the cylinder.



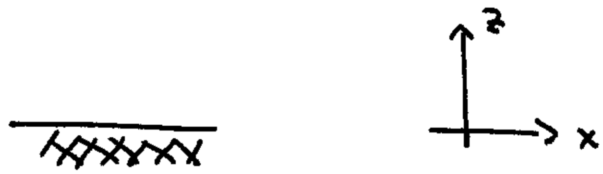
In this discussion, I have considered only structureless, incompressible fluids. If we add structure to a fluid, the new properties bring in new dimensionless numbers. For example, if we consider the ability of a fluid to transport heat, the diffusion constant for heat κ enters. Then there is a new dimensionless ratio, called the *Prandtl number*,

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

For small Prandtl number, heat is carried by diffusion; for large Prandtl number, by convection.

Finally, I would like to study the problem of an oscillating body in a viscous fluid. We will not have as complete a solution to this problem as we found early for a moving body in an ideal fluid. But we can understand what effect viscosity has on that solution.

Begin with the simplest problem, an oscillating wall bounding a viscous fluid



$$\overleftrightarrow{U(t)} = \mathrm{Re}\, U_0 e^{-i\omega t}$$

The motion of the wall induces a fluid flow in the $\hat{x}$ direction $v_x(t)$. We can look for a solution to the Navier-Stokes equation of the form

$$\vec{V} = V_x(z,t)\,\hat{x} \qquad\qquad p = p(z)$$

Here is the Navier-Stokes equation once again:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}$$

For the solution we seek, the $\hat{z}$ component is

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

so p is constant in this case. Also, automatically, $(\vec{v}\cdot\vec{\nabla})\vec{v} = 0$, so

$$\frac{\partial}{\partial t} v_x = \nu \nabla^2 v_x$$

Look for a solution

$$v_x = \mathrm{Re}\; a\, e^{-i\omega t + ikz}$$

The simplified Navier-Stokes equation implies

$$-i\omega a = -\nu k^2 a$$

The boundary condition at $z = 0$ is $v_x(0, t) = U(t)$; thus

$$a = U_0$$

Then the solution is

$$v_x = \mathcal{R}e \; U_0 \, e^{-i\omega t + ikz}$$

with

$$k = \sqrt{\frac{i\omega}{\nu}} = \pm \frac{1+i}{\sqrt{2}} \sqrt{\frac{\omega}{\nu}}$$

The velocities must go to zero far from the wall, so we must choose the positive square root. Then

$$v_x = \mathcal{R}e \left[U_0 \, e^{-i\omega t} \, e^{i\sqrt{\frac{\omega}{2\nu}} z} \right] \cdot e^{-\sqrt{\frac{\omega}{2\nu}} z}$$

This is a *transverse* oscillation of fluid that damps out as we go into the bulk. The characteristic size of the oscillating layer is

$$\delta = \sqrt{\frac{2\nu}{\omega}}$$

the *penetration depth.* The friction force on the surface per unit area is

$$F_x / \text{area} = \eta \left. \frac{\partial v_x}{\partial z} \right|_{z=0}$$

That is,

$$F_x / A = \eta \sqrt{\frac{\omega}{\nu}} \, \mathrm{Re} \, v_0 e^{-i\omega t} \frac{i-1}{\sqrt{2}}$$

$$= -\eta \sqrt{\frac{\omega}{\nu}} \, \mathrm{Re} \, v_0 \, e^{-i\omega t} e^{-i\pi/4}$$

or, finally,

$$F_x / A = -\eta \sqrt{\frac{\omega}{\nu}} \, v_0 \cos(\omega t + \pi/4)$$

The work done by the fluid onthe surface is

$$\frac{1}{\text{Area}} \frac{dE}{dt} = F(t) v(t) = -\eta \sqrt{\frac{\omega}{\nu}} \, U_0^2 \cos\left(\omega t + \frac{\pi}{4}\right) \cos \omega t$$

The average work over a cycle is computed from

$$\langle \cos(\omega t + \pi/4) \cos \omega t \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

which gives

$$\left\langle \frac{dE}{dt} \right\rangle / \text{Area} = -\frac{1}{2\sqrt{2}} \sqrt{\omega \eta \rho}\ v_0^2$$

Can we generalize these results to a body of arbitrary shape? In general, we can no longer ignore the $(\vec{v} \cdot \vec{\nabla})\vec{v}$ term. However, I will restrict this discussion to Stokes regime or the regime of low Reynolds number.

Consider the oscillations of a body of size ℓ. We can compare this length scale to the penetration depth computed for the problem of a large surface

$$\delta = \sqrt{\frac{2\nu}{\omega}}$$

There are two limiting cases

$$\delta \gg \ell \qquad \text{or} \qquad \delta \ll \ell$$

Since δ depends on the frequency ω, *slow* oscillations give $\delta \gg \ell$ while *fast* oscillatons (but still with small R) give $\delta \ll \ell$.

In the case of slow oscillations, we can treat the flow as approximately steady. Then we would compute the Stokes flow around the object and use this to determine the instantaneous drag force. For a sphere, the drag force at any point in the motion would be

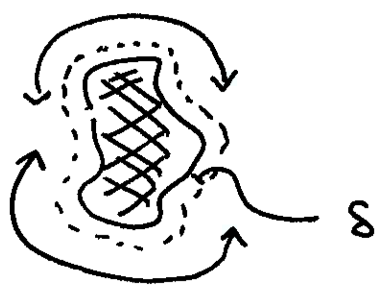
$$\vec{F}_D = -6\pi\eta R\, \vec{U}(t)$$

For a more complicated shape, the solution would be more involved. The only general statement we can make is that, by dimensional analysis,

$$\vec{F}_D = -C\eta R\,\vec{U}(t)$$

for some dimensionless constant C characteristic of the body.

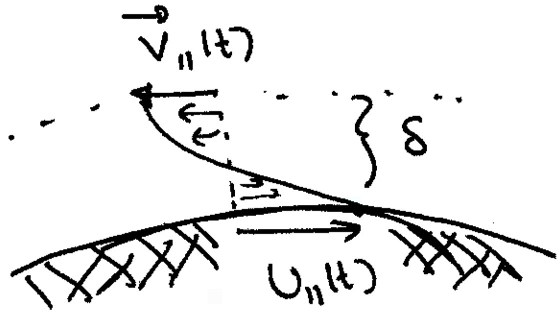
We can go further in the case of fast oscillations, large ω. In this case, the penetration depth is small and the transverse velocity induced by the viscous forces in the fluid die away outside a thin layer around the body. Outside this layer, we have a laminary flow obeying the boundary condition $\hat{n} \cdot \vec{v} = 0$ but with an arbitrary velocity parallel to the surface of the body. The difference between this transverse velocity and the velocity of the body begins to go to zero for distances from the body less than the penetration depth. This problem is then solved in the following way: Outside the thin layer, we have a potential flow, exactly the same one as in our earlier discussion with an ideal fluid.



Let the trangential velocity associated with this flow be

$$\Delta\vec{V}_t = \left[\vec{V}(t) - \vec{U}(t)\right]_{\parallel}$$

Near the surface, we have



and the fluid motion there is just what we computed for an oscillating plane.

This dissipation is thus confined to the surface layer. To compute it, we can borrow the formula derived in this section,

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{1}{2\sqrt{2}}\sqrt{\eta\omega\rho}\int d^2s \; \left|\Delta\vec{v}_{\parallel}\right|_0^2$$

where $\Delta\vec{v}_t\big|_0$ is the amplitude of the oscillation of $\Delta\vec{v}_t$ at the given point on the surface.