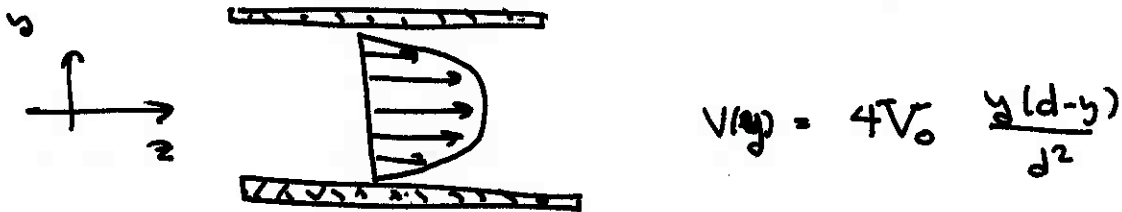


Fluid Instabilities (continued)

In the Couette problem, viscosity acts as a stabilizing influence. Reynolds already asked the question: Is it possible that a flow can be stable at $\nu = 0$ and be destabilized with $\nu \neq 0$. An example is given by 2-dimensional Poiseuille flow. This is the flow between parallel planes satisfying the boundary condition $\vec{v} = 0$ on the two planes



as discussed in an earlier lecture.

We can first analyze the stability of this flow with $\nu = 0$. The basic equations are

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p \quad \nabla \cdot \vec{v} = 0$$

These equations are solved by any flow of the form

$$\vec{v} = V(y) \hat{z} \quad p = p_0 = \text{const.}$$

This is similar to the result that we found for rotational flow. If $\nu = 0$, the layers of fluid can freely slip over one another. As in the Couette problem, I will study the stability of a particular solution, the solution to the problem with nonzero viscosity. In this case, we must also solve

$$\nu \nabla^2 \vec{v} = 0 \quad \vec{v} = 0 \text{ at } y=0, d$$

and then there is a single steady solution, Poiseuille flow.

To analyze the stability of this flow, expand to first order about this solution.

$$\vec{v} = V(y) \hat{z} + \delta \vec{v} \quad p = p_0 + \delta p$$

and keep terms of first order in small quantities. This gives

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right) \delta \vec{v} + \delta v_y \frac{\partial V}{\partial y} \hat{z} = -\frac{1}{\rho} \nabla \delta p$$

$$\nabla \cdot \delta \vec{v} = 0$$

We can solve the second equation as we did in the lecture on boundary layers, by introducing a stream function $\psi = \Psi + \delta \psi$. Here

$$\Psi = \Psi(y) \quad V = \frac{\partial \Psi}{\partial y}$$

gives the zeroth order solution, and for the perturbations

$$\delta v_z = \frac{\partial}{\partial y} \delta \psi \quad \delta v_y = -\frac{\partial}{\partial z} \delta \psi$$

Now break up $\delta\psi$ and δp into Fourier components. For simplicity, I will ignore variations in the flow along the \hat{x} direction that points into the page. It can be show that the modes of strongest instability are uniform in x . Then

$$\delta\psi = \psi(y) e^{-i\omega t + ikz} \quad \delta p = p(y) e^{-i\omega t + ikz}$$

The equations above become

$$-i(\omega - Vk) v_y = -\frac{p'}{\rho}$$

$$-i(\omega - Vk) v_z + v_y V' = -ik \frac{p}{\rho}$$

where

$$v_z = \psi' \quad v_y = -ik\psi$$

using the notation

$$\psi' = \frac{d}{dy} \psi \quad p' = \frac{d}{dy} p$$

Then we can solve for p ,

$$-ik \frac{p'}{p} = k(\omega - Vk)(-ik\psi)$$

$$\text{also} = \frac{d}{dy} [-i(\omega - Vk)\psi' - ikV'\psi]$$

$$= -i [\omega\psi'' - Vk\psi'' - V'k\psi' + kV'\psi' + kV\psi'']$$

and eliminate p to obtain

$$(\omega - Vk)(\psi'' - k^2\psi) + kV''\psi = 0$$

This is an eigenvalue equation that we must solve for a wavefunction ψ . The boundary condition \vec{v} is $v_y = 0$ on both boundaries. This is equivalent to $\psi = 0$ at $y = 0$ and $y = d$.

To find an instability, we must find an ω such that $\text{Im}\omega > 0$. Rayleigh found an argument that this is impossible for 2-dimensional Poiseuille flow. Here is the analysis. Rewrite the eigenvalue equation as

$$\psi'' - k^2\psi + \frac{V''}{\omega/k - V}\psi = 0$$

Multiply by $(-\psi^*)$ and integrate $\int_0^d dy$. Integrate the first term by parts. Then

$$\int_0^d dy \left(\left| \frac{d\psi}{dy} \right|^2 + k^2 |\psi|^2 - \frac{V''}{\omega/k - V} |\psi|^2 \right) = 0$$

If ω has an imaginary part, $\omega/k = \alpha + i\beta$. Then

$$\frac{1}{\omega/k - V} = \frac{\alpha - V - i\beta}{|\alpha - V|^2 + \beta^2}$$

Now, $V(z)$ and V'' are real and $|\psi|^2$ is real, so the imaginary part of this equation is

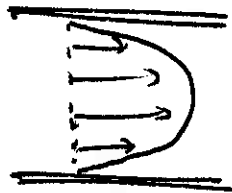
$$\beta \int_0^d \frac{V''}{|\alpha - V|^2 + \beta^2} |\psi|^2 = 0$$

For plane Poiseuille flow,

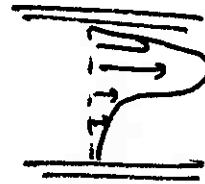
$$V'' = -8 \frac{V_0}{d^2}$$

so we can only obtain zero by setting $\beta = 0$. Then ω cannot have an imaginary part.

More generally, a 2-dimensional flow in a channel can be unstable only if $V(z)$ has an inflection point where V'' changes sign.



stable



unstable

When the effects of viscosity are added, the problem becomes much harder. The equation that result from Fourier reduction are generalized to

$$-i(\omega - Vk)v_y + D \frac{p}{\rho} = \nu (D^2 - k^2)v_y$$

$$-i(\omega - Vk)v_z + V'v_y + ik \frac{p}{\rho} = \nu (D^2 - k^2)v_z$$

where $D = d/dz$. After eliminating p , this is a fourth-order equation for ψ , and we must address this by imposing the full boundary condition $v_y = v_z = 0$, or $\psi = \psi' = 0$, at $y = 0$ and $y = d$. The resulting equation for ψ is the *Orr-Sommerfeld equation*

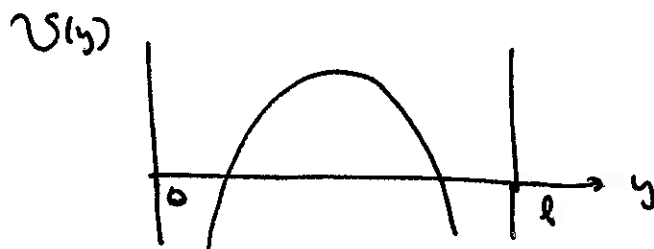
$$-i(\omega - Vk)(\psi'' - k^2\psi) - i k V''\psi = \nu (D^2 - k^2)\psi$$

Sommerfeld is indeed *the* Sommerfeld of Bohr-Sommerfeld quantization. His student Heisenberg worked on this equation for his Ph.D. but it was too hard for him to solve completely. He then turned to an easier problem—the invention of quantum mechanics. Much later, in the late 1940's, C. C. Lin worked out the basic features of the solution.

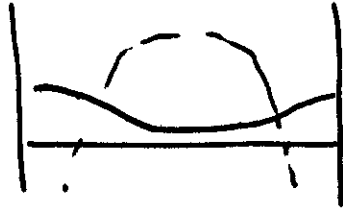
The easiest case is high Reynolds number, and even this is very subtle. Begin from the eigenvalue equation for the inviscid case, specialized to the case of Poiseuille flow,

$$-\frac{d^2}{dy^2}\psi + \left[k^2 - \frac{8V_0/d^2}{\left(\frac{4V_0}{d^2} - \omega/k\right)} \right]\psi = 0$$

This is literally a Schrödinger equation, with a potential of the form



There is a smooth solution at zero energy, but this solution does not satisfy the boundary condition



However, at large Reynolds number, the Orr-Sommerfeld equation also has a boundary layer solution. If we rescale

$$\bar{z} = \frac{z}{\Delta}$$

the leading terms as $\Delta \rightarrow 0$ are

$$-i\omega \frac{1}{\Delta^2} \frac{d^2}{d\bar{z}^2} \psi = \nu \frac{1}{\Delta^4} \frac{d^4}{d\bar{z}^4} \psi$$

so if $\Delta \sim \sqrt{\nu}$ or $\sqrt{1/R}$, these terms are in balance and give a solution near $y = 0$ of the form

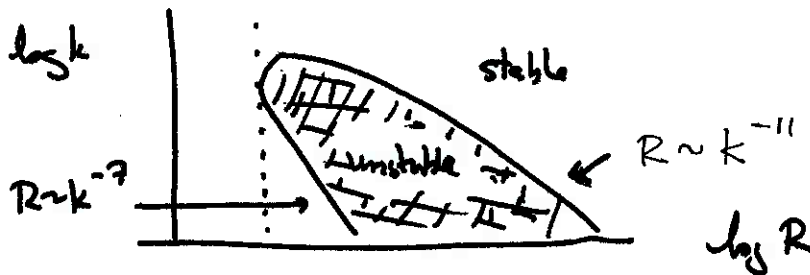
$$\psi \sim \exp \left[-\frac{1-i}{\sqrt{2}} \bar{z} \left(\frac{\omega}{\nu} \right)^{1/2} \right]$$

It is possible to use the WKB method to knit these solutions together to satisfy all of the boundary conditions. The WKB phase can generate $\text{Im} \omega > 0$. Lin found a

branch of the stability condition for fixed k . The dependence of the Reynolds number on k along this line is

$$R \sim k^{-11}$$

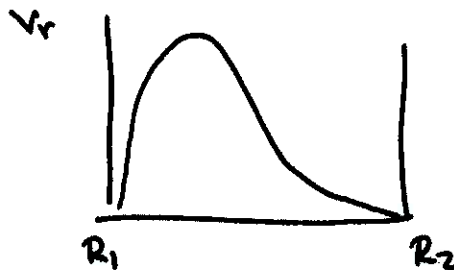
The more global picture in the (R, k) plane is



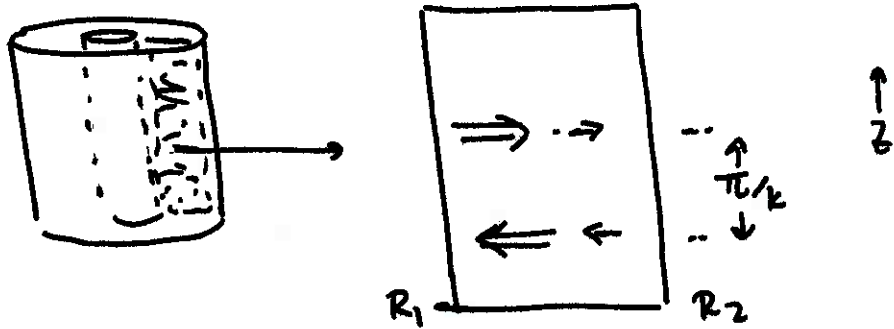
The dotted line shows the critical Reynolds number at which 2-dimensional Poiseuille flow becomes unstable.

Many more details of the analysis can be found in the book of Drazin and Reed, *Hydrodynamic Stability*.

Once the simple flow patterns become unstable, what happens next? In Couette flow, we found that the instability was associated with an eigenfunction of the form



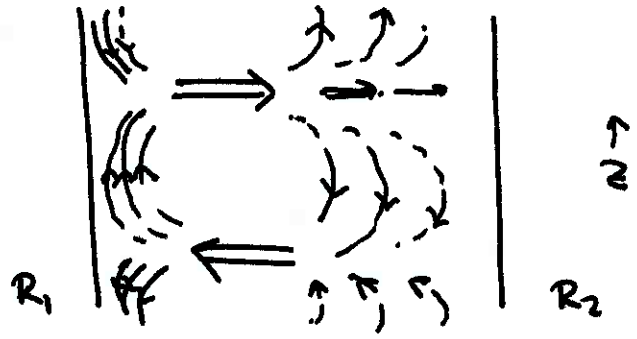
From this, we can draw the flow that results from turning on a perturbation of this form



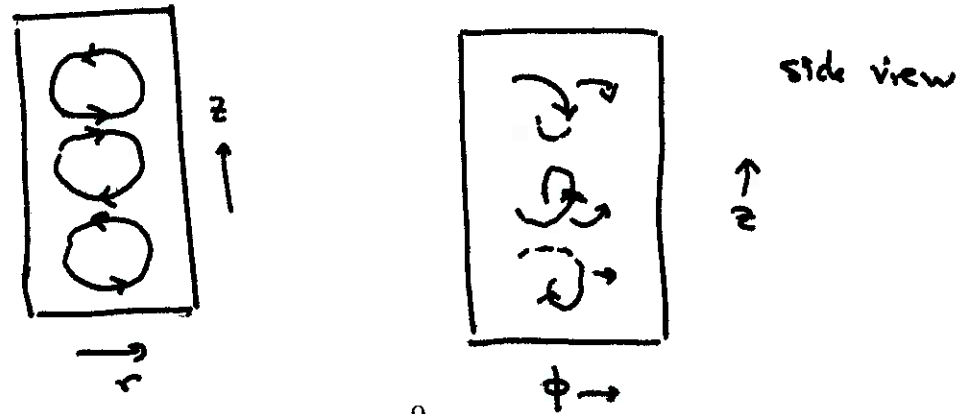
But, the fluid is incompressible, so there must also be a nonzero v_z . A result from our earlier analysis was

$$v_z = -\frac{i}{k} \frac{1}{r} \frac{d}{dr} (r v_r)$$

The $(-i)$ means that v_z is 90° out of phase with v_r in the oscillation in the \hat{z} direction. Including this vertical velocity, the picture becomes



The Couette flow, initially uniform in z , breaks up into rolls, called *Taylor vortices*,



At a higher value of the Reynolds number, this pattern becomes unstable to a more complex pattern, and there are additional thresholds as the Reynolds number is increased further.

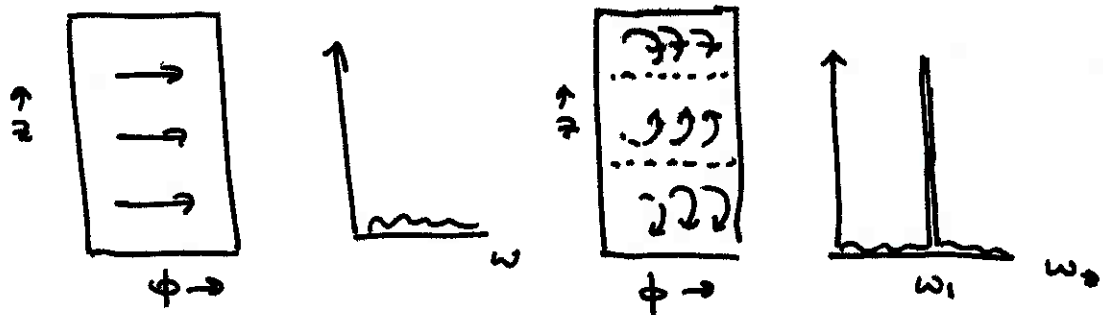
Eventually, the fluid reaches a regime in which it is undergoing complex, aperiodic motions,



This is called *turbulence*. Landau imagined that the transition from regular laminar flow to turbulence was one of steady accumulation of complexity. One might make an analogy to the transitions from a 2-atom system to a gas. In this picture, the fluid passes through a large number of successive instabilities. At each stage, the motion is regular but becomes increasingly complex and difficult to analyze. Eventually, the motion is indistinguishable from chaos.

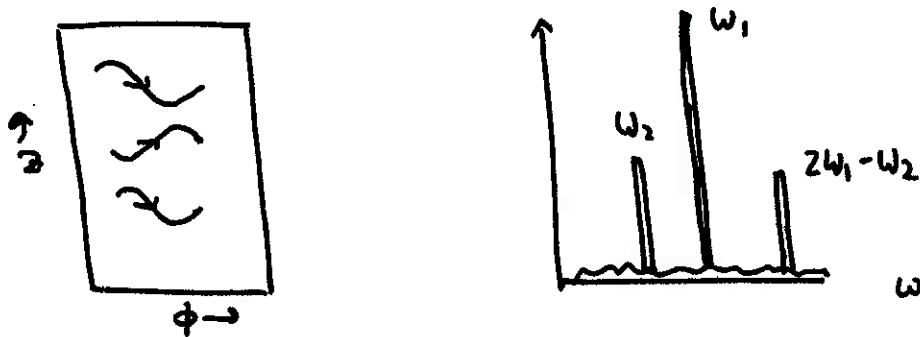
This hypothesis was questioned in the 1970's, and a number of alternative pictures were proposed. Very careful experiments were done to see which hypothesis actually applied in particular fluid systems. I will first describe the situation for Couette flow. A particularly beautiful study of this system can be found in Fenstermacher, Swinney, and Gollub, *J. Fluid Mechanics*, 94, 103 (1979). These authors set up a Couette flow with $\Omega_2 = 0$, $\Omega_1 \neq 0$ by rotating the inner boundary cylinder. They suspended polystyrene spheres in the water, scattered laser light from these spheres, and measured the local fluid velocities from the Doppler shifts. I will discuss the evolution of the Fourier spectrum in time of the velocity distribution as the Reynolds number is increased.

As Ω_1 is slowly increased from zero, the first two steps are very simple. If R_c is the critical Reynolds number for the first instability and ω_1 is the frequency of the periodic oscillation of the flow of the basic roll pattern.

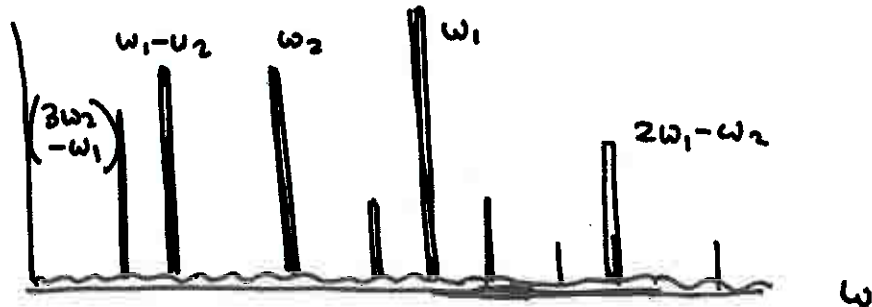


At $R/R_c \sim 10$, there is a second instability to a more complex roll pattern with a

second frequency ω_2



The frequencies ω_1 and ω_2 are incommensurate. In this case, the nonlinearity of the fluid equations will lead to perturbations at the combined frequencies $n\omega_1 + m\omega_2$. As Ω_1 is increased, the Fourier coefficients associated with these modes increase.



However, there is another transition at $R/R_c > 12$, above which a completely *aperiodic* component appears. The size of this component grows with R .



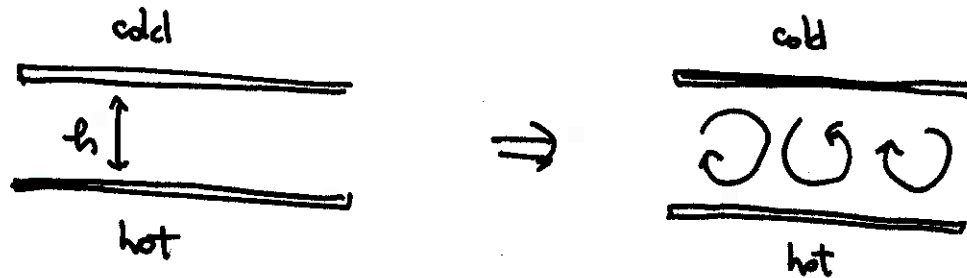
Eventually, by $R/R_c \sim 22$, the sharp peaks disappear and a broad, smooth frequency spectrum is all that remains.

There is an analogue of this behavior in a system of three ordinary differential equations studied by Lorenz in a problem arising in the dynamics of the atmosphere: E. N. Lorenz, J. Atmos. Sci. 20, 130 (1963). The Lorenz equations are

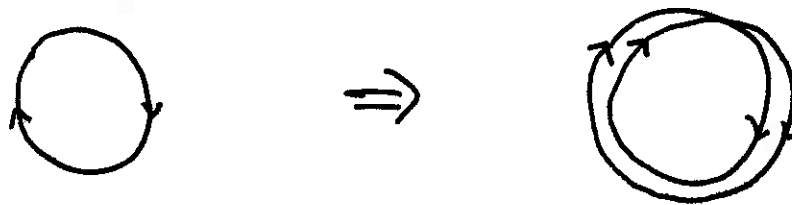
$$\begin{aligned}\dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned} \quad b, r, \sigma > 0$$

As r is increased, the solution $(x, y, z) = 0$ evolves through three successive instabilities. The final result is an *aperiodic* solution that is dense in a 2-dimensional subspace of the 3-dimensional phase space. This solution is called a *strange attractor*. Ruelle and Takens proposed this picture, with a finite number of instabilities leading to truly aperiodic behavior, as a general alternative to Landau's picture.

However, a different route to aperiodic turbulent behavior is found in different system with a sequence of fluid instabilities. Later in the course, we will study the *Rayleigh-Bénard instability* of fluid between two horizontal plates, heated from the bottom and cooled at the top.



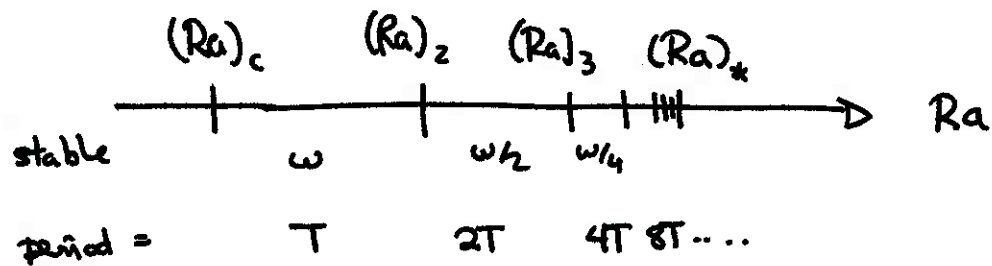
Libchaber and Maurer made a careful study of this system and showed that it reaches aperiodic behavior through a series of *period-doubling instabilities*. This route to chaos had been proposed in the 1970's by Feigenbaum (M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978)). At static situation has a transition to motion begins with a single frequency ω . At a second transition, there is an instability to a motion with frequency $\omega/2$. In essence, two adjacent cycles deform slightly from one another



At the next transition, the frequency halves again, and the period doubles, by a similar deformation. These transitions accumulate at a specific value of the Reynolds

number or another characteristic parameter. Beyond the limit point of the transitions, the motion is truly aperiodic.

In the Rayleigh-Bénard problem, the evolution is characterized by a dimensionless parameter Ra called the *Rayleigh number*. This number is proportion to the temperature gradient across the cell and inversely proportional to the diffusion constant for heat. The picture of the successive transitions is then



The instabilities accumulate according to

$$\frac{(Ra)_{i+1} - (Ra)_i}{(Ra)_i - (Ra)_{i-1}} \approx \frac{1}{\delta}$$

From the study of the *logistic equation*, the simple iteration

$$x_{i+1} = 4\lambda x_i (1 - x_i)$$

Feigenbaum had predicted that the ratio δ would have a universal value equal to

$$\delta = 4.669\dots$$

Libchaber's very beautiful experiments (Libchaber, Laroche, and Fauve, J. de Physique Lett. 43, L211 (1982)) measured

$$\delta = 4.4 \pm 0.1$$

for the periodic doubling cascade of the Rayleigh-Bénard flows in Mercury.

Are these the only routes from uniform laminar flow to chaos? This is still an open question for research.