Shock Waves (continued)

In this lecture, I will continue my discussion of shock waves. I will first derive some further consequences of the general structure of shock discontinuities presented in the previous lecture.

We can understand the Rankine-Hugoniot conditions derived in the previous lecture in a more explicit way by writing these conditions for the specific case of an ideal gas equation of state. We saw in our earlier discussion of compressible fluids that, for an ideal gas, the enthalpy can be written as

$$h = c_p \frac{k_B T}{m} = \frac{\gamma}{\gamma - 1} pV = \frac{c^2}{\gamma - 1}$$

It is illuminating to plug this relation into the consequence of the Rankine-Hugoniot conditions

$$h_1 - h_2 + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

The equation becomes

$$\gamma(p_1 V_1 - p_2 V_2) + \frac{\gamma - 1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

which can be rewritten as

$$\frac{\gamma}{2}\left[-(V_1 + V_2)(p_2 - p_1) + (V_1 - V_2)(p_2 + p_1)\right] + \frac{\gamma - 1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

$$\gamma(V_1 - V_2)(p_2 + p_1) - (V_1 + V_2)(p_2 - p_1) = 0$$

$$V_1\left[(\gamma + 1)p_1 + (\gamma - 1)p_2\right] = V_2\left[(\gamma - 1)p_1 + (\gamma + 1)p_2\right]$$

Then, finally,

$$\frac{V_2}{V_1} = \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}$$

From this, we can see easily our previous conclusion that $p_2 > p_1$ implies $V_1 > V_2$. In addition,

$$\frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1}$$

which we can now represent as

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}$$

Our earlier relation

$$j^2 = \frac{p_2 - p_1}{V_1 - V_2}$$

now becomes

$$j^2 = (p_2 - p_1)\,\frac{(\gamma-1)p_1 + (\gamma+1)p_2}{2(p_2 - p_1)V_1}$$

or

$$j^2 = \frac{(\gamma-1)p_1 + (\gamma+1)p_2}{2V_1}$$

Now $j_1 V_1 = v_1$, $j_2 V_2 = v_2$, so we can find the entering and exiting velocities as

$$v_1^2 = \frac{1}{2} V_1 [(\gamma-1)p_1 + (\gamma+1)p_2]$$

$$v_2^2 = \frac{1}{2} V_1 \frac{[(\gamma+1)p_1 + (\gamma-1)p_2]^2}{(\gamma-1)p_1 + (\gamma+1)p_2}$$

It is instructive to compare these to the formulae for the speed of sound on the front and back sides of the shock,

$$c_1^2 = \gamma p_1 V_1 \qquad c_2^2 = \gamma p_2 V_2$$

We see directly that, if $p_2 > p_1$, then

$$v_1^2 > c_1^2 \qquad v_2^2 < c_2^2$$

Another useful way to write the jump conditions is to introduce the Mach number

$$M = \frac{v}{c} \qquad M_1^2 = \frac{v_1^2}{c_1^2}$$

Then the above formulae give

$$\frac{V_2}{V_1} = \frac{v_2}{v_1} = \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2} = \frac{(\gamma-1)M_1^2 + 2}{(\gamma+1)M_1^2}$$

or

$$\frac{v_2}{v_1} = \frac{V_2}{V_1} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2}$$

An important limit is that of a strong shock, in which $v_1 \gg c_1$, or $M_1 \gg 1$. In this limit

$$\frac{v_2}{v_1} = \frac{V_2}{V_1} \rightarrow \frac{\gamma-1}{\gamma+1}$$

and also

$$\frac{v_1^2}{c_1^2} \rightarrow \frac{\frac{1}{2}V_1(\gamma+1)p_2}{\gamma p_1 V_1} = \frac{1}{2}\frac{\gamma+1}{\gamma}\frac{p_2}{p_1}$$

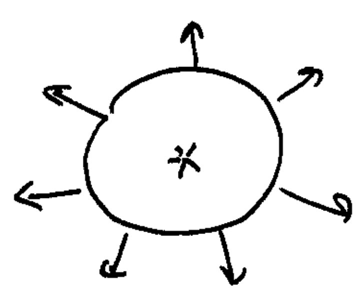
so

$$\frac{p_2}{p_1} \rightarrow \frac{2\gamma}{\gamma+1} M_1^2$$

As an application of these formulae, I will now analyze the problem of the growth of the shock wave from an explosion. Imagine that, in a region of gas at normal density ρ_0 and pressure p_0, we set off an explosion that suddenly releases an energy E in a very small region. The explosion creates a shock wave that expands radially

ρ_0 p_0

$\vec{v} = 0$



The shock wave expands at supersonic speed, gathering up and accelerating the gas in the atmosphere that it passes through. As the shock transfers its energy to the gas, it slows systematically. I will now work out the rate of expansion as a function of time.

Let $R(t)$ be the radial position of the shock wave at time t. The velocity of expansion of the shock is $\dot{R}(t)$. Now we can estimate the mass of the gas swept up by the shock

$$\sim \rho_0 R^3$$

the fluid velocity just behind the shock

$$\sim \dot{R} \sim \frac{R}{t}$$

and the kinetic energy in the shock

$$\sim \rho_0 R^3 \cdot (\dot{R})^2$$

This last quantity can be equated to the original energy E liberated by the explosion. This gives

$$E \sim \rho_0 R^5 / t^2$$

Thus we find the somewhat nontrivial relation

$$R(t) = \left(\frac{E}{\kappa \rho_0} \right)^{1/5} t^{2/5}$$

where κ is a dimensionless constant of order 1. The speed of the shock slows as

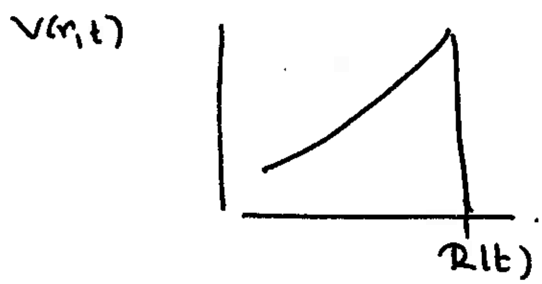
$$\dot{R}(t) \sim t^{-3/5}$$

as the shock transfers its energy to a greater volume of gas.

It is possible to build a much more complete picture of the expanding shock by looking for a similarity solution to the equations of fluid dynamics that depends on a single scaling variable. A properly dimensionless combination is the variable

$$\xi = \frac{r}{R(t)} = \frac{r\, t^{-2/5}}{(E/\kappa\rho_0)^{1/5}}$$

I will now try to represent the fluid variables for a spherically symmetric expanding system $v(r,t)$, $\rho(r,t)$, and $p(r,t)$ in terms of ξ. I will take the point $\xi = 1$ to be the position of the shock discontinuity. The form of $v(r,t)$ is then

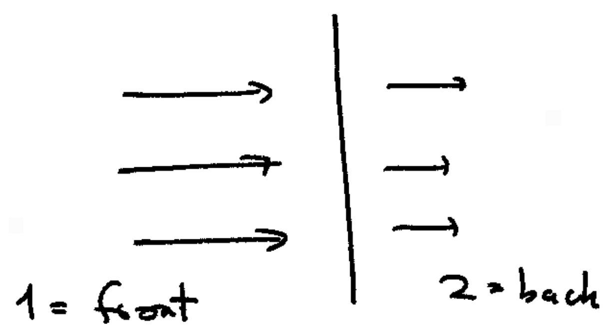


with a sharp transition to $v = 0$ at $\xi = 1$. We can write the three variables as

$$v(r,t) = v_x\, \overline{v}(\xi)$$
$$\rho(r,t) = \rho_x\, \overline{\rho}(\xi)$$
$$p(\xi,t) = p_x\, \overline{p}(\xi)$$

where the scales v_x, ρ_x, p_x are to be determined, and $\overline{v}(\xi)$, $\overline{\rho}(\xi)$, $\overline{p}(\xi)$ all have the value 1 at $\xi = 1$.

The boundary values v_x, ρ_x, and p_x at the discontinuity can be determined from the Rankine-Hugoniot conditions. To work this out, we must first understand which side of the discontinuity is the front and which side is the back. In the discussion above, we analyzed the the jump conditions in the frame in which the shock was at rest. In that frame, we found, for a strong shock and the assumption of the ideal gas equation of state

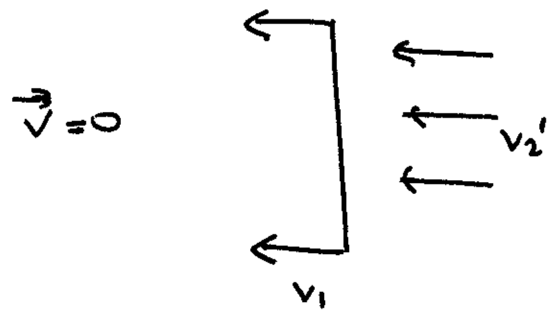


$$\frac{v_2}{v_1} = \frac{\gamma-1}{\gamma+1}$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\frac{v_1^2}{c_1^2}$$

We are viewing the blast wave in the frame in the frame in which the external atmosphere is at rest and the shock is moving at supersonic speed. That picture is related to the one above by a boost to the left by the velocity v_1, so that the fluid to the left comes to rest.



Somewhat counterintuitively, the outside of the shock is the *front* side and the interior of the expanding region is the *back* side.

Since density and pressure are left invariant by Galilean boosts, the relations between the front side and back side pressure in the atmosphere frame are exactly those in the figure above. The front side density and pressure are ρ_0 and p_0, so we conclude that the density and pressure just inside the shock discontinuity are

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0$$

$$p = \frac{2\gamma}{\gamma+1}\frac{\dot{R}^2}{c_0^2}\cdot p_0 = \frac{2}{\gamma+1}\rho_0\dot{R}^2$$

The velocity just inside the shock is related to v_2 above by

$$v_2' = \dot{R} - v_2 = \dot{R} - \dot{R}\frac{\gamma-1}{\gamma+1}$$

and therefore

$$v_2' = \frac{2}{\gamma+1}\dot{R}$$

Thus, the scaling forms of the velocity, density, and pressure can be written

$$v = \frac{2}{\gamma+1}\dot{R}\,\bar{v}(\xi)$$

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0\,\bar{\rho}(\xi) \qquad \dot{R} = \frac{2}{5}\left(\frac{E}{\kappa\rho_0}\right)^{1/5} t^{-3/5}$$

$$p = \frac{2}{\gamma+1}\rho_0\dot{R}^2\,\bar{p}(\xi)$$

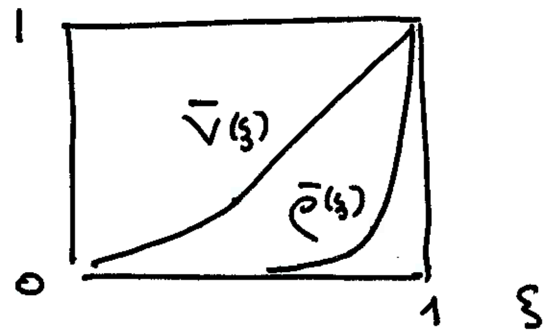
These formulae can now be plugged into the fluid dynamics equations with spherical symmetry

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial}{\partial r}v + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)(p/\rho^\gamma) = 0$$

turning these equations into a set of three ordinary differential equations determining $\overline{v}(\xi)$, $\overline{\rho}(\xi)$, $\overline{p}(\xi)$. A detailed solution can be found in Landau and Lifshitz. The form of the solution is as one would expect



Finally, the order 1 constant κ is determined by equating the total energy computed from the solution to the initial energy E.

This solution was discovered independently by G. I. Taylor and L. Sedov – both of whom we have encountered already in this course – for a very particular application. IN the fall of 1945, pictures were published of the first atomic bomb explosion at Alamogordo. The photographs showed that the expansion of the blast wave was described by the formula

$$R(t) = 60\,m \cdot \left(\frac{t}{1\,ms} \right)^{2/5}$$

Taylor and Sedov were able to infer from this information the yield E of the explosion, which was at that time a military secret:

$$E = 1.5 \times 10^{14}\,J = 30 \text{ kton TNT}$$

The pictures of the explosion are shown in the textbook of Blandford and Thorne, Fig. 16.17. Sometimes it is useful to know more than a little about fluid mechanics.

I will now turn to another important topic in supersonic flow, the theory of the supersonic nozzle. This theory will turn out to have an interesting astrophysical application.

We return to the problem of laminar flow in a pipe. Now I would like to consider the pipe to have a varying cross-sectional area. I will study the problem of whether it is possible to have a subsonic flow enter the pipe and, because the pipe gets narrower, have a supersonic flow emerge from the other end.

I will treat the fluid as ideal, ignoring viscosity and heat exchange. Then Bernoulli's theorem applies. Along each streamline, in steady flow, the quantity

$$\mathcal{B} = h + \frac{1}{2}v^2$$

is a constant, which we may set equal to h_0, the enthalpy at a place well upstream where $v = 0$. The equation of steady flow is

$$\vec{v} \cdot \vec{\nabla} \, \vec{v} = -\frac{1}{\rho} \vec{\nabla} p$$

This implies

$$v \, dv = -\frac{1}{\rho} dp = -\frac{c^2}{\rho} d\rho$$

or

$$\frac{d\rho}{dv} = -\frac{\rho v}{c^2}$$

The particle number current

$$j = \rho v$$

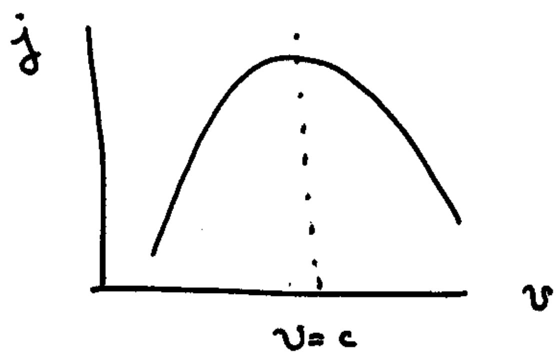
then satisfies

$$\frac{d\dot{j}}{dv} = \frac{d}{dv}(\rho v) = \rho - \rho\frac{v \cdot v}{c^2}$$

or

$$\frac{d}{dv}\dot{j} = \rho\left(1 - v^2/c^2\right)$$

This is an odd relation. As v increases, ρ decreases. The net mass flow increases until v reaches the local value of the speed of sound. Above this value, the decrease in ρ dominates and j decreases.



Denote the point in the flow where $v = c$ as $*$. The local value of c varies as ρ varies, so the point $*$ is defined by the condition

$$v_* = c_*$$

where both $v(x)$ and $c(x)$ must be determined by the solution of the fluid equations.

We can work out the formulae in detail for an ideal gas. From our general discussing of compressible gases,

$$h = \frac{c^2}{\gamma - 1}$$

Then the Bernoulli equation is

$$\frac{c^2}{\gamma-1} + \frac{1}{2}v^2 = \frac{c_0^2}{\gamma-1}$$

or

$$c^2 = c_0^2 - \frac{1}{2}(\gamma-1)v^2$$

Putting $v_* = c_*$, we can solve for c_*,

$$c_* = c_0\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}$$

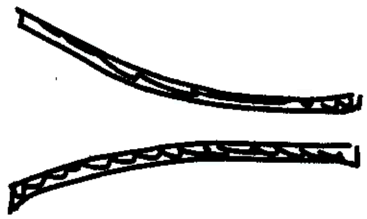
Since $c \sim \rho^{(\gamma-1)/2}$ or $\rho \sim c^{2/(\gamma-1)}$,

$$\rho_* = \rho_0\left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)}$$

Then

$$\dot{j}_* = \rho_* c_* = \dot{j}_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

Now apply these consideratios to 1-dimensional steady flow in a *nozzle*, a pipe of varying cross-section,



I will write $S(x)$ for the cross-sectional area. The mass flow through the nozzle is

$$Q = \rho v S = j S$$

This is *constant* as a function of x. Then

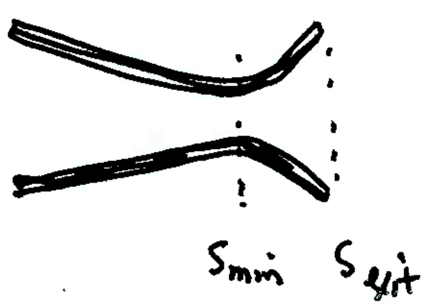
$$j(x) = \frac{Q}{S(x)}$$

You might think that, by exerting enough pressure on the gas in the nozzle, we can set up a flow that becomes supersonic down the pipe. For a pipe of continually decreasing cross section, however, this idea is not consistent with the equations. The maximum value of j must occur at the point where S is minimal. But

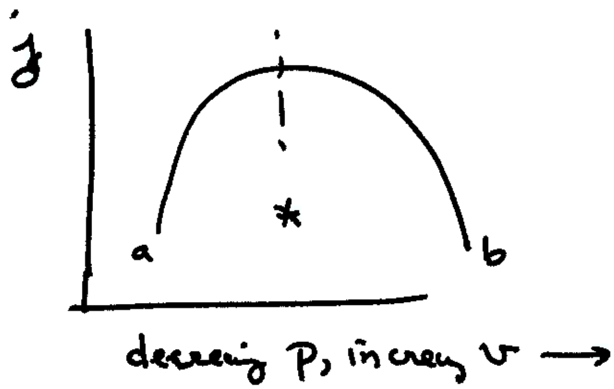
$$j_{max} = j_*$$

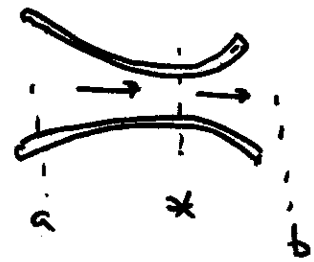
So either the flow is never supersonic or $v = c$ only at the end of the pipe.

However, we can achieve supersonic flow using another shape, the *de Laval nozzle*, which narrows and then widens again.

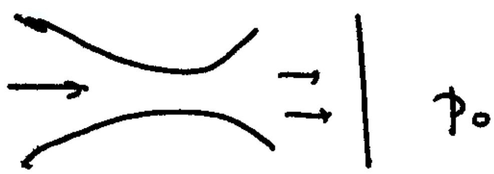


The minimal cross section is S_{min}, and this is smaller than the cross section at the exit S_{exit}. We can set up a flow in which $j = j_*$ is reached where $S = S_{min}$. To the right of S_{min}, j decreases, but we can go onto the branch of $j(v)$ where the flow is supersonic. The pressure decreases steadily from left to right





The the right of the exit, the flow should return to subsonic flow. The transition from the supersonic flow in the latter half of the nozzle to the subsonic flow beyond must involve a shock wave. If the pressure at the exit is greater than atmospheric pressure, the shock will be to the right of the exit



If we increase p_0 or decrease the high pressure at the front of the nozzle, the shock wave will come back into the nozzle

Finally, when the shock wave reaches $*$, it disappears and the flow becomes subsonic everywhere.

The de Laval nozzle shape is used in rocket engines. The force transmitted to the surroundings, the *thrust*, is given by

$$\text{Thrust} = T^{xx} \cdot S_{exit}$$

$$= (p + \rho v^2)_{exit} \cdot S_{exit}$$

To the right of $*$ it is a good approximation to ignore p and consider the stress as dominated by the ρv^2 term. Then

$$\text{Thrust} = Q \cdot v_{exit}$$

or

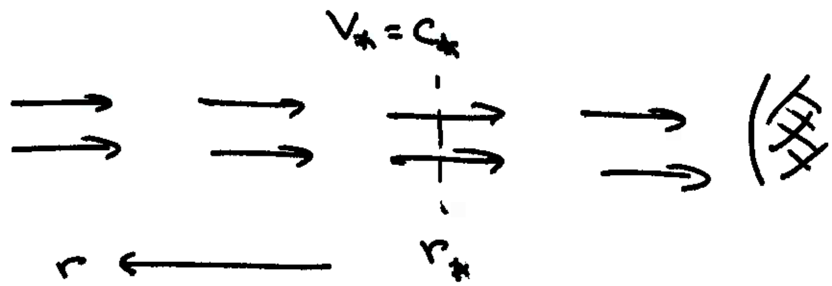
$$\text{Thrust} = Q \cdot c_{exit} \cdot M_{exit}$$

where M_{exit} is the Mach number of the gas at the exit

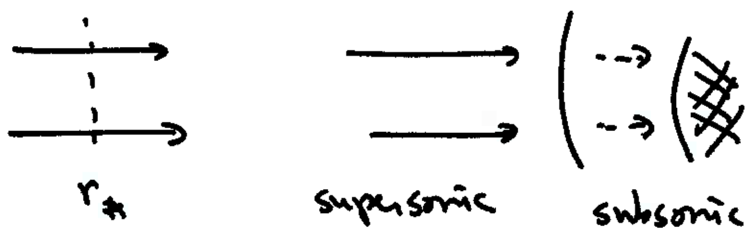
$$M_{exit} = \frac{v_{exit}}{c_{exit}}$$

An optimal design matches the exit p to p_0 and maximizes M_{exit}.

This theory has a direct application to the theory of a spherical cloud of gas falling into a black hole or other compact astrophysical object. This problem was analyzed by Bondi in 1952. He realized that the infalling gas can only become supersonic by passing through a transition similar to that in a de Laval nozzle. The infall rate is then determined by the point in the flow at which the infall speed of the gas is equal to the local speed of sound,



What happens next depends on the nature of the compact object. If this object is a neutron star, the gas must obey $v_r = 0$ at the surface of the star, and thus a shock wave must be set up just above the star.



If the object is a black hole, the gas will continue to accelerate as it falls into the center. In either case, however, the rate at which the object accretes gas is independent of these considerations and depends only on physics at the radius r_* where $v_* = c_*$ and the gas just becomes supersonic.

We can readily work at the formulae relevant to this case. We assume a steady, spherically symmetric flow

$$\vec{v} = -v(r)\,\hat{r}$$

Then the equation of mass conservation

$$\vec{\nabla}(\rho\vec{v}) = 0 \quad \text{or} \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0$$

implies

$$\rho \frac{\partial v}{\partial r} + v \frac{\partial \rho}{\partial r} + 2 \frac{\rho v}{r} = 0$$

The Navier-Stokes equation is

$$(\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{1}{\rho} \vec{\nabla} p - \frac{GM}{r^2} \hat{r}$$

or, for the spherically symmetry flow,

$$v \frac{\partial}{\partial r} v = - \frac{c^2}{\rho} \frac{\partial \rho}{\partial r} - \frac{GM}{r^2}$$

Combining this with the mass equation, we find

$$v \left(- \frac{v}{\rho} \frac{\partial \rho}{\partial r} - 2 \frac{v}{r} \right) = - \frac{c^2}{\rho} \frac{\partial \rho}{\partial r} - \frac{GM}{r^2}$$

or

$$(v^2 - c^2) \frac{1}{\rho} \frac{d\rho}{dr} = \frac{GM}{r^2} - 2 \frac{v^2}{r}$$

Thus, the radius at which the flow becomes supersonic obeys the condition

$$v_*^2 = c_*^2 = \frac{GM}{2 r_*}$$

Now apply the Bernoulli equation

$$\mathcal{B} = \frac{1}{2}v^2 + h + \Phi = \text{const.}$$

together with the formula for the enthalpy of an ideal gas

$$h = \frac{c^2}{\gamma - 1}$$

At infinity, where the gas velocity is zero,

$$h_\infty = \frac{c_\infty^2}{\gamma - 1}$$

At the radius r_*,

$$\mathcal{B} = \frac{1}{2}c_*^2 + \frac{c_*^2}{\gamma-1} - \frac{GM}{r_*}$$

$$= \left(\frac{1}{2} + \frac{1}{\gamma-1} - 2\right) c_*^2 = \frac{5-3\gamma}{2(\gamma-1)} c_*^2$$

Equating these quantities, we find

$$c_*^2 = \frac{2}{5-3\gamma} c_\infty^2$$

Then, since

$$r_* = \frac{GM}{2 c_*^2}$$

we find an explicit expression for r_* in terms of properties of the gas and the mass of the compact object

$$r_* = \frac{5-3\gamma}{4} \frac{GM}{c_\infty^2}$$

We are now in a position to compute the mass accretion rate $\dot{M}$ of the compact object. This is given fundamentally by

$$\dot{M} = 4\pi r_*^2 \cdot c_* \cdot \rho_*$$

Putting in the values above

$$\dot{M} = 4\pi \left(\frac{5-3\gamma}{4}\right)^2 \frac{(GM)^2}{c_\infty^4} \left(\frac{2}{5-3\gamma}\right)^{1/2} c_\infty \rho_*$$

Also,

$$\frac{\rho_*}{\rho_\infty} = \left(\frac{c_*}{c_\infty}\right)^{2/\gamma-1} = \left(\frac{2}{5-3\gamma}\right)^{1/\gamma-1}$$

This gives, finally, the *Bondi accretion rate*

$$\dot{M} = \left(\frac{2}{5-3\gamma}\right)^{\frac{\gamma+1}{2(\gamma-1)}} 4\pi r_*^2 c_\infty \rho_\infty$$

or

$$\dot{M} = \pi \left(\frac{5-3\gamma}{2}\right)^{-\frac{5-3\gamma}{2(\gamma-1)}} \rho_\infty \frac{(GM)^2}{c_\infty^3}$$

The dependence on γ in this formula is odd. Bondi accretion works straightforwardly only if we have a mixture of gases for which $\gamma > \frac{5}{3}$. The value $\gamma = \frac{5}{3}$ is that for a pure monatomic gas, for example, pure atomic Hydrogen. As $\gamma \to \frac{5}{3}$, the value of the radius r_* in this nonrelativistic calculation tends to 0. However, general relativity makes the gravitational field stronger at short distances and actually causes r_* to be fixed at a radius still well outside the compact object. The general relativity prediction for Bondi accretion is actually close to the limit of the above formula for $\gamma \to \frac{5}{3}$. A thorough discussion of Bondi accretion using general relativity can be found in the textbook of Shapiro and Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars.*