Mar. 11

Magnetohydrodynamics

For our final topic, I will discuss the addition to fluid dynamics of electric and magnetic fields. This adds profound complication to the the subject, since the complexity of the configuration of field can be as great as the complexity of the flow pattern, and these two configurations can have strong nonlinear interactions. In principle, one could plan a whole course around this topic. Here I have only one lecture, so I will discuss only the most basic results.

To describe the interaction of flows with electromagnetic fields, I will model the fluid as a *conductor* that transmits macroscopic currents. This models has as its equations the Navier-Stokes equation plus Maxwell's equations. This composite macroscopic theory is called *magnetohydrodynamics* or MHD. This theory is applied to flows of liquid metals and to flows of highly ionized gases (*plasmas*). The application to liquid metals is clear, but more plasmas one must consider the question of whether a purely macroscopic approach makes sense. Plasmas can be highly diffuse, with densities of less than 10^{11} particles/cm^3 and macroscopic mean free paths for scattering. However, the new feature of MHD is the coupling of particles or local current to long-ranged fields which then act back on the particles. If this is the primary interaction, then even if there are few collisions, a plasma can behave as a macroscopic fluid. In this lecture, I will put this question aside and use macroscopic equations even for diffuse plasmas.

I will now write the basic equations of MHD and discuss some suitable approximations. We can begin with Maxwell's equations. I will use CGS units as in Jackson. Then

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

In these equations, and throughout this lecture, I use ρ_e for the electric charge density, reserving ρ for the fluid mass density. However, I will use the symbol $\vec{j}$ to denote the electric charge current and not the mass current $\rho\vec{v}$.

To close the equations, we need a constituitive equation that relates $\vec{j}$ back to the imposed electric and magnetic fields. For a fluid *at rest*, we should use Ohm's law

$$\vec{J} = \sigma \vec{E}$$

where σ is the conductivity of the fluid. In this lecture, I will take σ to be a scalar constant that responds instantaneouly to the applied fields; this approximation is appropriate to a non-relativistically moving fluid with a short mean free path. Still, Ohm's law must be modified in a moving fluid. We can see this in two different ways. First, the charged particles in a fluid moving through a nonzero $\vec{B}$ field feel a Lorentz force, so the above equation must be modified to

$$\vec{J} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

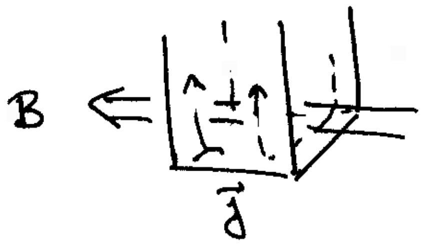
Alternatively, we could derive this equation by boosting to the frame in which the fluid is at rest. The $\vec{E}$ field in that frame is

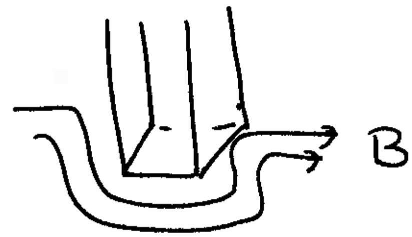
$$\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{B}$$

to first-order accuracy in the boost.

The current induced by the Lorentz force produce an effect that is one of the hallmarks of MHD. Think about what happens when we attempt to move a conductor into a magnetic field. The Lorentz force induces currents

which then create a $\vec{B}$ field that cancels the original one.





The result is that the $\vec{B}$ field is repelled from the conductor. By a similar logic, a magnetic field originally present in a conductor is frozen into the conductor as the conductor moves.

For the application of Maxwell's equations to fluid dynamics, it is useful to make some further approximation and rearrangement. First, I should not that, in the equation for $\vec{\nabla} \times \vec{B}$, the term with $\partial \vec{E}/\partial t$ can be neglected in situations in which the fluid moves nonrelativistically. If V and L are characteristic velocities and lengths in the fluid, and$\vec{E}$ and $\vec{B}$ are in balance according to Faraday's law,

$$\vec{\nabla} \times \vec{E} \sim \frac{E}{L} \sim -\frac{1}{c}\frac{\partial B}{\partial t} \sim \frac{1}{c}\frac{BV}{L}$$

then

$$\frac{1}{c}\frac{\partial E}{\partial t} \sim \frac{E}{L}\frac{V}{c} \sim \frac{V^2}{c^2}\frac{B}{L} \sim \frac{V^2}{c^2} \cdot \vec{\nabla} \times \vec{B}$$

Then Ampere's law reduces to a simple form with no time derivatives

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$$

With this approximation, we can write

$$\vec{J} = \sigma\left(\vec{E} + \frac{\vec{V}}{c} \times \vec{B}\right)$$

and

$$\vec{E} = -\frac{\vec{v}}{c}\times\vec{B} + \frac{1}{\sigma}\vec{j} = -\frac{\vec{v}}{c}\times\vec{B} + \frac{c}{4\pi\sigma}\vec{\nabla}\times\vec{B}$$

Take the curl of this equation. We find

$$\frac{\partial B}{\partial t} = -c\,\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{c^2}{4\pi\sigma}\left[-\vec{\nabla}\times(\vec{\nabla}\times\vec{B})\right]$$

and

$$-\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) + \nabla^2\vec{B}$$

For an incompressible fluid

$$\vec{\nabla}\times(\vec{v}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{v} + \vec{v}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) - (\vec{v}\cdot\vec{\nabla})B - \vec{B}(\underbrace{\vec{\nabla}\cdot\vec{v}}_{=0})$$

We then obtain

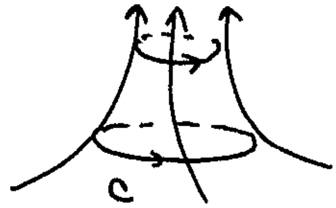
$$\frac{\partial\vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{v} + \frac{c^2}{4\pi\sigma}\nabla^2\vec{B}$$

This is just the same equation that we found earlier in the course for the *vorticity* of an incompressible fluid. The left-hand side of the equation is

$$\frac{D}{Dt}\vec{B} = \frac{\partial\vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})B$$

that is, simple convection of the $\vec{B}$ field. The first term on the right-hand side courses the $\vec{B}$ field to spin up in a converging flow. If the last term is ignored, this equation preserves the magnetic flux carried by a ring of fluid

$$\frac{d}{dt}\left(\int_C d^2s\, \hat{n}\cdot\vec{B}\right) = 0$$



The last term gives *diffusion* of $\vec{B}$ across the flow. This diffusion, which is a dissipative effect, goes to zero in a fluid of high conductivity. The diffusion constant can be thought of as a *magnetic viscosity*

$$D_M = \frac{c^2}{4\pi\sigma}$$

We can also define a *magnetic Reynolds number*

$$R_M = \frac{VL}{D_M} = \frac{4\pi\sigma}{c^2} VL$$

This number R_M is often large in MHD flows, and the regime of large R_M naturally contains *magnetic turbulence.*

Blandford and Thorne give a useful table of values of R_M and the magnetic diffusion time τ_M

	L (m)	V (m/sec)	D_M (m^2/sec)	R_M	$\tau_M = L^2/D_M$
liquid Hg	0.1	0.1	1	10^{-2}	10^{-2} sec
laboratory plasma	1	100	10	10	10^{-1} sec
Earth's core	10^7	0.1	1	10^6	10^{14} sec $= 10^7$ yr
Interstellar Plasma	10^{17}	10^3	10^3	10^{17}	10^{31} sec.

In the latter two cases, turbulent diffusion can be significant, leading to much shorter diffusion times.

Along with the above flow equation for $\vec{B}$, which follows from Maxwell's equations, we must consider the Navier-Stokes equation for the velocity field. Ignoring *viscosity* but adding the effect of electric and magnetic fields, and still assuming incompressibility, this equation becomes

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\,\vec{v} \right] = -\vec{\nabla}p + \rho_e \vec{E} + \frac{\vec{j}}{c}\times\vec{B}$$

In a good conductor, charge will not accumulate, and we can ignore ρ_e. Under this approximation, and using

$$\frac{\vec{j}}{c} = \frac{1}{4\pi}\vec{\nabla}\times\vec{B}$$

the field-dependent terms become

$$\frac{\vec{j}}{c}\times\vec{B} = -\frac{1}{4\pi}B\times(\vec{\nabla}\times\vec{B}) = -\frac{B^i}{4\pi}\vec{\nabla}B^i + \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

This term falls nicely into the form of a contribution to the stress tensor

$$-\nabla^k T_B^{ik}$$

with

$$T_B^{ik} = \frac{B^2}{8\pi}\delta^{ik} - \frac{B^i B^k}{4\pi}$$

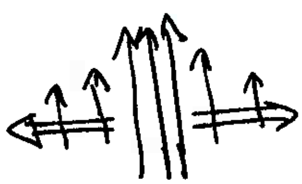
For example, if $\vec{B} \parallel \hat{z}$, the stress tensor has the form

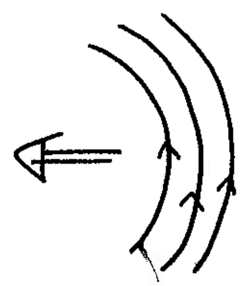
$$T_B^{ik} = \frac{1}{8\pi} \begin{pmatrix} B^2 & & \\ & B^2 & \\ & & -B^2 \end{pmatrix}$$

In directions normal to the field lines, there is a *magnetic pressure*

$$\text{pressure} = \frac{B^2}{8\pi}$$

More generally, magnetic field configurations exert forces on the fluid that tend to spread out and straighten magnetic field lines





The full Navier-Stokes equation under the approximations just listed is

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p - \vec{\nabla}\frac{B^2}{8\pi} + \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

The corresponding energy equation is

$$\frac{\partial}{\partial t}\rho_\epsilon + \vec{\nabla}\cdot\vec{J}_\epsilon = 0$$

with

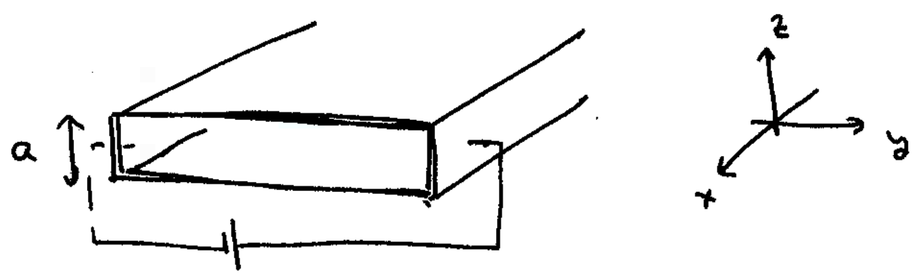
$$\rho_\epsilon = \rho\left(\frac{v^2}{2} + u\right) + \frac{B^2}{8\pi}$$

$$\vec{j}_\epsilon = \rho\vec{v}\left(\tfrac{1}{2}v^2 + h\right) + \frac{c}{4\pi}\vec{E}\times\vec{B}$$

Finally, the entropy equation must be modified to include Ohmic heating

$$\rho T\left[\frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s\right] = \kappa\nabla^2 T + \frac{j^2}{\sigma}$$

To illustrate these equations, I will now apply them to a very simple MHD flow called *Hartmann flow.* This is a generalization of Poiseuille flow, adding the influence of electric and magnetic fields. We have already studied the flow of a viscous fluid between parallel plates, driven by pressure. Consider such a flow in the $\hat{x}$ direction, with plates oriented normal to $\hat{z}$. Add to this the possibility of an external $\vec{B}$ field in the $\hat{z}$ direction normal to the plates, and the possibility of an $\vec{E}$ field in the $\hat{y}$ direction that might be induced by putting a voltage across the system.



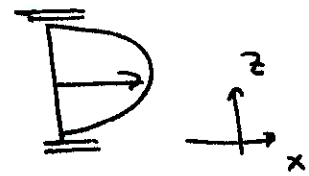
I will assume that the box is much wider than it is high and ignore the boundary condition at the vertical boundaries of the box.

The fluid in the box can be pumped by a pressure gradient,

$$-\vec{\nabla}p = G\hat{x}$$

For $\vec{E} = \vec{B} = 0$, we have Poiseuille flow

$$v = \frac{G}{2\eta} z(a-z)$$



Now turn on nonzero E_y, B_z. We will find that we must also allow a nonzero B_x (along the flow) to obtain the full solution. For a steady flow with $\vec{v} \parallel \hat{x}$, $|\vec{v}|$ function of z only, the Navier-Stokes equation is

$$0 = -\vec{\nabla} p + \frac{\vec{j} \times \vec{B}}{c} + \eta \frac{\partial^2}{\partial z^2} \vec{v}$$

with

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right) \qquad j_y = \sigma \left(E_y - \frac{v}{c} B_z \right)$$

In components,

$$G = -\frac{dp}{dx} = -\left[\frac{\sigma}{c} \left(E_y - \frac{v}{c} B_z \right) B_z + \eta \frac{\partial^2 v}{\partial z^2} \right]$$

and

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\frac{\sigma}{c} \left(E_y - \frac{v}{c} B_z \right) B_x$$

After we find B_x, we can integrate the last equation to find the dependence of pressure on z.

The first of the three equations can be reorganized into

$$\frac{d^2}{dz^2} V - \frac{M^2}{a^2} V = -\frac{G}{\eta} - \frac{M^2}{a^2} c\left(\frac{E_y}{B_z}\right)$$

where

$$M^2 = \frac{\sigma B_z^2 a^2}{\eta c^2}$$

The parameter M is a dimensionless variable called the *Hartmann number.* This number is essentially the ratio of magnetic and viscous forces. The equation illustrates several different effects. First, it says that, for $E_y = 0$, we can balance a pressumre gradient against MHD forces. The $\vec{v} \times \vec{B}$ induced by the pressure gradient produces a $\vec{j}$ that brakes the flow even without viscosity. Second, it says that an E_y app;lied to the plasma will drive a nonzero v even without a pressure gradient. This is a *magnetic pump.*

For $M \gg 1$, we expect the form of the velocity distribution $v(z)$ to be different from that in Poiseuille flow. Indeed, the solution to the equation with $v(z) = 0$ at $z = 0, a$ is

$$V(z) = V_0 \left(1 - \frac{\cosh \frac{M}{a}(z - a/2)}{\cosh \frac{M}{2}} \right)$$

where V_0 is proportional to G or E_y. This expression goes over into that for Poiseuille flow for $M \ll 1$. However, for $M \gg 1$, it shows a boundary layer of thickness

$$\Delta = a M^{-1/2} = \left[\frac{\eta}{\frac{\sigma}{c^2} B_z^2} \right]^{1/2}$$

Finally, we can solve for B_x. The equation

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

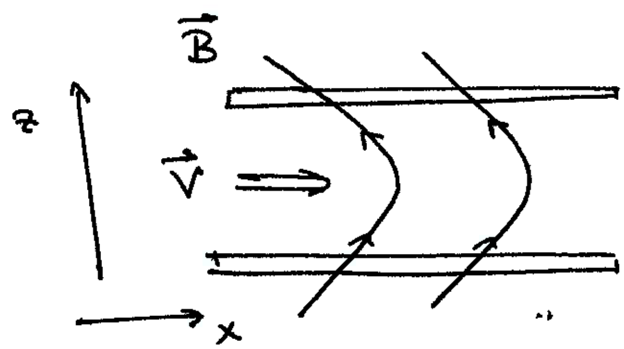
gives

$$\frac{\partial}{\partial z} B_x = \frac{4\pi}{c} j_y = \frac{4\pi}{c} \sigma \left(E_y - \frac{v}{c} B_z \right)$$

Integrating,

$$B_x = \int_{a/2}^{z} dz \, \frac{4\pi}{c} \sigma \left(E_y - \frac{v(z)}{c} B_z \right)$$

I have assumed symmetrical boundary conditions with $\vec{B}$ vertical in the center of the plasma. For the case $E - y = 0$, a flow driven by pressure, we have $B_x < 0$ for $z > a/2$,



Then, quite explicitly, the magnetic field lines are dragged by the moving fluid.

I will now discuss the theory of waves in plasmas. A compressible plasma supports sound waves, and it should also support other types of oscillations involving the interaction of fluid motion with an external electromagnetic field. In this discussion, I will only consider relatively low frequency waves. You are familiar with *plasma oscillations*, which occur at nonzero frequency in the microwave region as the result of charge and current oscillations. This effect requires that we go beyond Ohm's law and take advantage of the fact that the response of charged particles to a time-varying electric field is not instantaneous. However, plasmas support a variety of nontrivial waves within the domain of MHD. I will not study these waves systematically.

To find the simplest examples, consider an incompressible conducting fluid in a magnetic field. Then $\vec{\nabla} \cdot \vec{v} = 0$, and there is an order-1 background field

$$\vec{B} = \mathbb{B}\hat{z} , \quad \vec{\mathbb{B}}$$

Write the MHD equations, with zero viscosity,

$$\frac{\partial \vec{B}}{\partial t} + \vec{v}.\vec{\nabla}\vec{B} = \vec{B}.\vec{\nabla}\,\vec{v} \qquad \frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \frac{1}{4\pi\rho}\vec{B}\times(\vec{\nabla}\times\vec{B})$$

and linearize about the solution with $\vec{v} = 0$ and $\vec{B}$ as above. The situation is most easily analyzed in Fourier modes. As we have done previously in this course, write the fluctuations of $\vec{v}$, $\vec{B}$, and p as

$$\delta\vec{v} = \vec{v}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \qquad \text{etc.}$$

where the Fourier coefficients are complex numbers. The linearized equations are then

$$i\vec{k}\cdot\vec{v} = 0$$

$$-i\omega\vec{B} = (\vec{\mathbb{B}}\cdot i\vec{k})\,\vec{v}$$

$$-i\omega\vec{v} = -\frac{i\vec{k}p}{\rho} - \frac{1}{4\pi\rho}\vec{\mathbb{B}}\times(i\vec{k}\times\vec{B})$$

Combining the last two equations, we find

$$-\omega^2\vec{v} = -\frac{1}{4\pi\rho}\vec{\mathbb{B}}\,(i\vec{k}\times\vec{v})(i\vec{k}\cdot\mathbb{B}) - \frac{\omega\vec{k}}{\rho}p$$

Since $\vec{k}\cdot\vec{v} = 0$, terms parallel to $\vec{k}$ are irrelevant. Once we are in the subspace perpendicular to $\vec{k}$, we see from this equation that $\vec{v}$ is also perpendicular to $\vec{\mathbf{B}}$. Represent these constraints by writing

$$\vec{v} = A\,\vec{k}\times\vec{\mathbf{B}}$$

Now we can plug this formula into the equation above

$$A\omega^2\,\vec{k}\times\vec{\mathbf{B}} = -\frac{A}{4\pi\rho}\,\vec{\mathbf{B}}\times(\vec{k}\times(\vec{k}\times\vec{\mathbf{B}}))\;\vec{k}\cdot\vec{\mathbf{B}}$$

$$= -\frac{A}{4\pi\rho}\left[\vec{k}\;\underbrace{\vec{\mathbf{B}}\cdot(\vec{k}\times\vec{\mathbf{B}})}_{=0} - \vec{k}\cdot\vec{\mathbf{B}}(\vec{k}\times\vec{\mathbf{B}})\right]\vec{k}\cdot\vec{\mathbf{B}}$$

or

$$A\omega^2(\vec{k}\times\vec{\mathbf{B}}) = \frac{A}{4\pi\rho}(\vec{k}\cdot\vec{\mathbf{B}})^2\,(\vec{k}\times\vec{\mathbf{B}})$$

The coefficient of A is a dispersion relation for a propagating wave

$$\omega^2 = \frac{(\vec{k}\cdot\vec{\mathbf{B}})^2}{4\pi\rho}$$

If θ is the angle between $\vec{k}$ and $\vec{\mathbf{B}}$, then

$$\omega = \frac{\mathbf{B}\cos\theta}{\sqrt{4\pi\rho}}\,k$$

The waves that appear here are called *Alfvén waves.* The perturbation $\delta\vec{B}$ is perpendicular to $\vec{\mathbf{B}}$, so this is an oscillation in the direction of the magnetic field

The restoring force is provided by the MHD force that straightens magnetic field lines. The inertia is provided by the mass of the fluid.

If the fluid is compressible, additional modes appear. (At the very least, we must find sound waves.) To find these, add a possible density perturbation

$$\rho = \rho_0 + \delta\rho = \rho_0 + \rho\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

related to δp by

$$\delta p = c_s^2\, \delta\rho \qquad\qquad c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

The equation of mass conservation is now

$$-i\omega\rho + i\vec{k}\cdot\vec{v}\,\rho_0 = 0$$

Then the linear approximation to the Navier-Stokes equation becomes

$$-i\omega\rho_0\vec{v} = -i\, c_s^2\, \vec{k}\rho - \frac{1}{4\pi}\vec{B}\times(i\vec{k}\times\vec{B})$$

We also need the equation for $\vec{B}$, and here we should restore the term that we dropped above by assuming $\vec{\nabla}\cdot\vec{v} = 0$,

$$\vec\nabla\times(\vec v\times\vec B) = (\vec B\cdot\vec\nabla)\vec v - (\vec v\cdot\vec\nabla)\vec B - \vec B(\vec\nabla\cdot\vec v)$$

The linearized $\vec B$ equation is then

$$-i\omega\vec B = (\vec B\cdot i\vec k)\vec v - \vec B(i\vec k\cdot\vec v)$$

For modes of $\vec v$ orthogonal to both $\vec k$ and $\vec{\mathbf{B}}$, the previous results still apply. To find more general solutions, combine the two equations above into

$$\omega^2\vec v = \frac{1}{4\pi\rho}\left[\vec B\times(i\vec k\times\vec v)\,(i\vec k\cdot\vec B) - \vec B\times(i\vec k\times\vec B)\,i\vec k\cdot\vec v\right]$$
$$+\ c_s^2\,\vec k\ \vec k\cdot\vec v$$

or

$$\omega^2\vec v = \frac{1}{4\pi\rho}\left[(\vec k\cdot\vec B)^2\vec v - \vec k(\vec k\cdot\vec B)\,\vec B\cdot\vec v + \vec k\ \vec k\cdot\vec v\ B^2\right.$$
$$\left. -\ \vec B\ \ \vec k\cdot\vec B\ \ \vec k\cdot\vec v\right] + c_s^2\,\vec k(\vec k\cdot\vec v)$$

Now dot with $\vec k$

$$\omega^2(\vec k\cdot\vec v) = \frac{1}{4\pi\rho}\left[\cancel{(\vec k\cdot\vec B)^2} + k^2B^2 - \cancel{(\vec k\cdot\vec B)^2}\right]\vec k\cdot\vec v + c_s^2k^2(\vec k\cdot\vec v)$$
$$-\ \frac{1}{4\pi\rho}\,k^2\ \vec k\cdot\vec B\ \ \vec B\cdot\vec v$$

and with $\vec{\mathbf{B}}$

$$\omega^2\, \vec{B}\cdot\vec{v} = \frac{1}{4\pi\rho}\left[\cancel{(\vec{k}\cdot\vec{B})^2 \vec{B}\cdot\vec{v}} - \cancel{(k\cdot B)^2}\, \vec{B}\cdot\vec{v} \right]$$

$$+ \frac{1}{4\pi\rho}\left[B^2\, \cancel{\vec{k}\cdot\vec{B}}\, \vec{k}\cdot\vec{v} - B^2\, \cancel{\vec{k}}\cdot\vec{B}\ \ \vec{k}\cdot\vec{v} \right] + c_s^2\, k\cdot\vec{B}\ k\cdot\vec{v}$$

$$= c_s^2\ \vec{k}\cdot\vec{B}\ \vec{k}\cdot\vec{v}$$

Combining the equations, we obtain a fourth-order equation

$$\omega^4(\vec{k}\cdot\vec{v}) = \left[\omega^2 k^2 \left[\frac{B^2}{4\pi\rho} + c_s^2 \right] - \frac{k^2 (\vec{k}\cdot\vec{B})^2 c_s^2}{4\pi\rho} \right] (\vec{k}\cdot\vec{v})$$

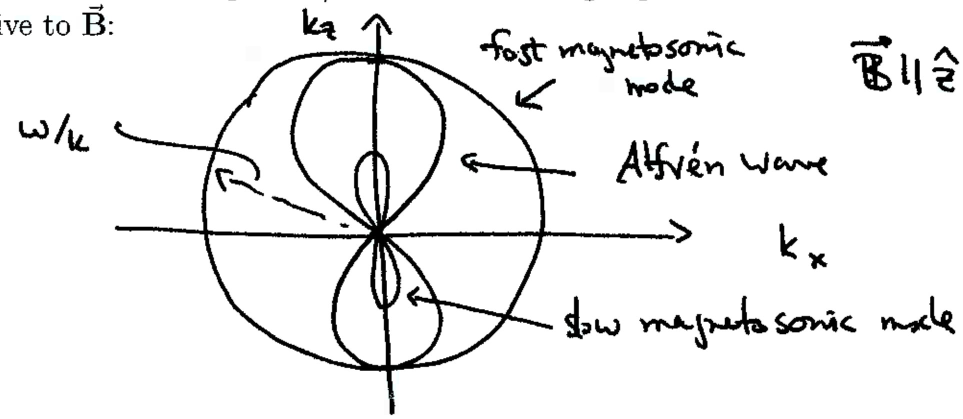
or

$$\left[\left(\frac{\omega}{k} \right)^2 \right]^2 - \left(\frac{\omega}{k} \right)^2 \left(\frac{B^2}{4\pi\rho} + c_s^2 \right) + c_s^2 \frac{(\hat{k}\cdot\vec{B})^2}{4\pi\rho} = 0$$

This equation has two solutions

$$\left(\frac{\omega}{k} \right)^2 = \frac{1}{2}\left\{ \left(\frac{B^2}{4\pi\rho} + c_s^2 \right) \pm \left[\frac{B^2}{4\pi\rho} + c_s^2 - \frac{4 c_s^2 B^2 \cos^2\theta}{4\pi\rho} \right]^{1/2} \right\}$$

These two modes are mixtures of the sound wave and a mode involving compression of the $\vec{B}$ field. The wave speed ω/k has the following dependence on the orientation of $\vec{k}$ relative to $\vec{B}$:



The modes at small ω, k are a sound wave and an Alfvén wave.

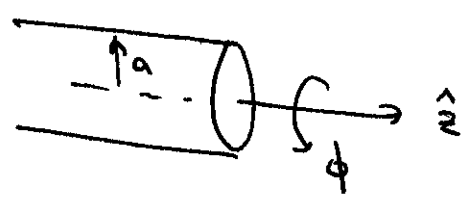
An important topic in plasma physics is the question of the *confinement* of plasmas by magnetic fields. One of the dreams of a certain segment of the physics community has been to develop controlled nuclear fusion as a power source. This requires maintaining an ionized Hydrogen/Deuterium plasma at high temperature for an extended period of time to allow the fusion reaction to take place. Such a plasma cannot touch the walls of a container, so the plasma must be maintained in a compact region by electromagnetic fields. This problem has given rise to a fascinating but very complex literature on time-independent plasma flows and their instabilities.

To discuss the basic notions of this subject, I will begin from the equations of *magneto-hydrostatics*, that is, time-independent configurations of plasma with $\vec{v} = 0$. The static Navier-Stokes equation is

$$0 = -\vec{\nabla} p + \frac{\vec{j} \times \vec{B}}{c}$$

This equation implies that $\vec{\nabla} p$ is orthogonal to both $\vec{j}$ and $\vec{B}$. The surfaces of constant p are called *isobars*; in particular, the surface of the plasma is an isobar. We see that $\vec{j}$ and $\vec{B}$ must be oriented parallel to this surface.

For the confinement of a cylindrical tube of plasma, there are two especially simple configurations. The first is the *z-pinch*, with $\vec{j} \parallel \hat{z}$, $\vec{B} \parallel \phi$.



We can analyze this assuming that the magnitudes of $\vec{j}$ and $\vec{B}$ depend only on r

$$\vec{j} = j_z(r)\,\hat{z} \qquad \vec{B} = B_\phi(r)\,\hat{\phi}$$

In the equation involving p, only the radial component is nontrivial

$$\frac{\partial p}{\partial r} = -\frac{j_z B_\phi}{c}$$

The relation

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

implies

$$\frac{1}{r} \frac{\partial}{\partial r}(r B_\phi) = \frac{4\pi}{c} j_z$$

The right-hand side of the magnetohydrostatic equation is

$$-\frac{j_z B_\phi}{c} = -\frac{1}{4\pi} \frac{B_\phi}{r} \frac{\partial}{\partial r}(r B_\phi)$$

so if

$$\frac{\partial p}{\partial r} = -\frac{1}{4\pi} \frac{B_\phi}{r} \frac{\partial}{\partial r}(r B_\phi)$$

we have a solution. The form of $B_\phi(r)$ depends on the penetration of B_ϕ and j_z into the plasma, which is determined by the conductivity. For low conductivity,

$$\underset{\text{inside}}{j_z = \text{const.}} \qquad B_\phi = \frac{2\pi}{c} j_z r \qquad p = \frac{\pi j_z^2}{c} (a^2 - r^2)$$

with $p = 0$ at the surface at $r = a$ and

$$\frac{\partial}{\partial r}\left(p + \frac{B^2}{4\pi}\right) = 0 \qquad \text{inside}$$

inside the plasma. For high conductivity, $j_z(r)$ is nonzero only in a small region on the surface of the cylinder. Across this region, p and B_ϕ have the discontinuities

$$\Delta p = -\Delta\left(\frac{B^2}{8\pi}\right)$$

so that

$$p = \left.\frac{B^2}{8\pi}\right|_{r=a^+} \qquad j_z = B = 0 \qquad \text{inside}$$

If I is the total current, the B field just outside the plasma is

$$B = \left.\frac{2I}{cr}\right|_{r=a^+}$$

so in the latter case the pressure inside is

$$p = \frac{I^2}{2\pi c^2 a^2}$$

We can put in some numbers for a plasm at the temperature needed for a controlled fusion reactor

$$k_B T \sim 10\,\mathrm{keV} \sim 10^{8}\,{}^\circ\mathrm{K} \qquad n = 10^{15}/\mathrm{cm}^3$$

Then, for

$$a \sim 1\,\mathrm{cm} \qquad I \sim 10^{5}\,\mathrm{Amp.}$$

we find

$$p = n k_B T \sim 14\,\mathrm{atm}$$

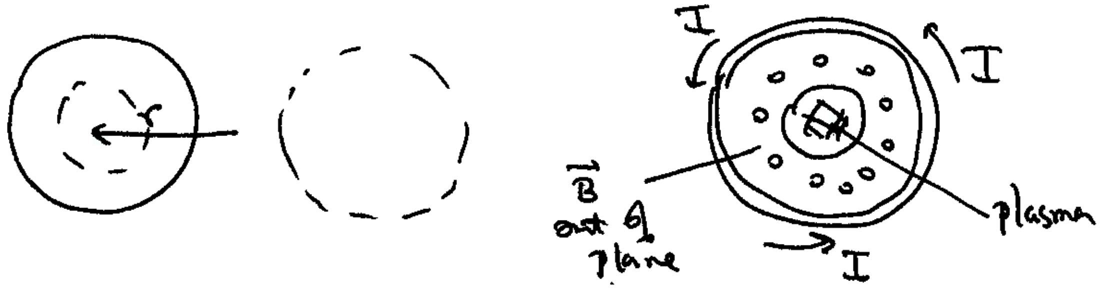
The other simple configuration is the *θ-pinch*, with

$$\vec{B} \parallel \hat{z} \qquad \vec{j} \parallel \hat{\phi} \quad (\text{or } \hat{\theta})$$

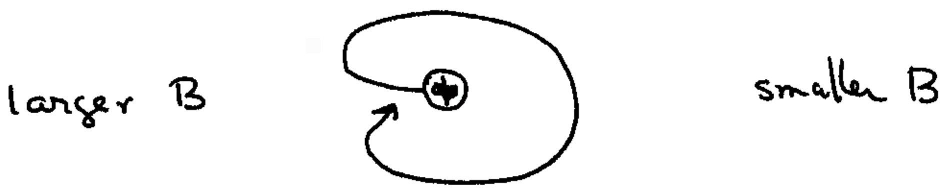
Again, if the plasma is a very good conductor, $\vec{j}$ will be a surface current. Then, for confinement

$$p_{\text{inside}} = \left(B^2/8\pi\right)_{\text{outside}}$$

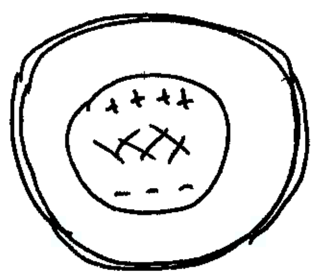
Of course, it is not sufficient simply to confine a plasma to a cylinder, because if the cylinder is not infinite in length, particles can escape out the ends. To ameliorate this problem, we can consider winding the cylinder into a torus,



However, this does not quite work. Because $\int d\vec{x}cdot\vec{B}$ around the torus is independent of r, being equal to the total current I flowing around, $\vec{B}$ is somewhat weaker at large r. This leads to an awkward effect at the single-particle level. Cyclotron orbits in the B field have smaller radii at larger B. This causes particles to move preferentially normal to $\vec{\nabla}B$. For positive charges

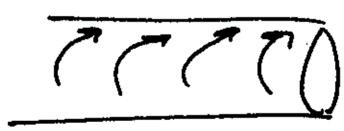


and for negative charges, the motion is in the opposite direction. Then the plasma develops charge separation



which leads to a macroscopic instability.

In the early 1950's, two solutions were proposed for this problem. Both entailed having current in the plasma that rotate around the cylinder and randomize their orientation.



Lyman Spitzer proposed a way to do this with external magnets, giving a device called the *stellarator*. Andrei Sakharov proposed driving the currents with external electric fields, giving a device called the *tokomak*. The tokomak, which has higher symmetry, was more successful. In the 1990's, the TFTR tokomak at Princeton achieved plasma confinement for several seconds and observed thermonuclear ignition; see R. J. Hawryluk et al., Physics of Plasmas, 5, 1577 (1998). The next step is a $ 10 B international facility called ITER that is now under construction in Cadarache, France.

Much more analysis is required to analyze an actual device and prove that it confines plasma stably. Even in the approximation of an infinite cylinder, the simple pinches analyzed above have fluid-mechanical instabilities that much be controlled by modifying the arrangements of currents and plasma velocities. As our final topic in plasma physics, I will analyze the fluid-mechanical instability of the simple z-pinch.

I will work in the limit $\sigma \to \infty$ where the current becomes a singular layer on the surface of the cylinder. The interior of the plasma is field-free. The solution to the magnetohydrostatic equations has a $\vec{B}$ field in the $\hat{\phi}$ direction, related to the total current by

$$B_\phi = \frac{2I}{cr}$$

and a pressure in the fluid that balances the external magnetic pressure

$$p = \left.\frac{B^2}{8\pi}\right|_{r=a^+}$$

Now assume a small perturbation of this situation. The new position of the surface will be

$$R = a + \eta(z,\phi,t)$$

The fluid velocity will be

$$\delta\vec{v}\,(r, z, \phi, t)$$

We can go to a Fourier representation in t, z, and ϕ. The dominant instability will be ϕ-independent. Then

$$\eta = \eta\, e^{-i\omega t + ikz} \qquad \delta\vec{v} = \vec{v}(r)\, e^{-i\omega t + ikz} \qquad \delta p = p(r)\, e^{-i\omega t + ikz}$$

with $v_\phi = 0$. If we treat the plasma as incompressible, the equations of fluid dynamics take the linearized form

$$\vec{\nabla}\cdot\vec{v} = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dr}(r v_r) + ik v_z = 0$$

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{\nabla}p \quad \Rightarrow \quad -i\omega\rho v_r = -\frac{d}{dr}p$$

$$-i\omega\rho v_z = -ikp$$

These give three equations in three unknowns. We can eliminate v_z, v_r in terms of p,

$$\frac{1}{r}\frac{d}{dr} r v_r = \frac{-i}{\omega\rho}\frac{1}{r}\frac{d}{dr} r\frac{d}{dr}p = -ik v_z = -i\frac{k^2}{\omega\rho}p$$

so that finally

$$\frac{1}{r}\frac{d}{dr} r\frac{d}{dr}p - k^2 p = 0$$

This is a modified Bessel equation. The solution regular at $r = 0$ is

$$p = A\, I_0(kr)$$

Then

$$v_r = \frac{-i}{\omega\rho} A \frac{d}{dr} I_0(kr) = \frac{-ik}{\omega\rho} A\, I_1(kr)$$

The relation between v_r and ζ is

$$\delta v_r \Big|_{r=a} = \frac{d\zeta}{dt} \quad \text{or} \quad v_r(a) = -i\omega\zeta$$

so

$$\zeta = \frac{k}{\omega^2\rho} A\, I_1(ka)$$

Finally, look at the boundary condition on the surface,

$$\delta B = -\zeta \frac{2I}{ca^2}$$

which implies

$$\delta\left(\frac{B^2}{8\pi}\right) = -\frac{1}{4\pi}\frac{\zeta}{a} B^2 = -\frac{k\, B^2}{4\pi a \omega^2 \rho} A\, I_1(ka)$$

The internal pressure at the surface must balance this, so that

$$p(a) = A I_0(ka) = -\frac{k}{4\pi\rho\omega^2} A I_1(ka)$$

From this, we find the dispersion relation for small oscillations

$$\omega^2 = -\frac{kB^2}{4\pi a\rho}\frac{I_1(ka)}{I_0(ka)}$$

But

$$I_0(ka),\ I_1(ka) > 0$$

so we see that

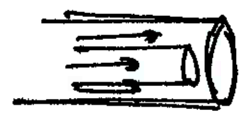
$$\omega^2 < 0$$

The solution is unstable. The instability grows faster for larger k or smaller wavelength.

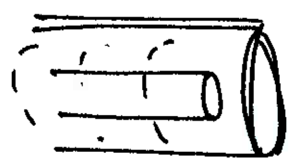
This instability of a confined plasma is known as the *sausage instability*

It is one of a catalogue of instabilities of cylindrical or toroidally confined plasmas. In the above analysis, we assumed that the small fluctuations were axisymmetric. However, there are also non-axisymmetric perturbations that are unstable, for example, the *kink instability*

In order to confine a plasma, we must engineer features that stabilize these instabilities, for example, adding an axial magnetic field



or using image fields from a conducting container



We have now come to the end of the course. We started our discussion with simple smooth fluid flows. But, in the course of the term, we have seen that the partial differential equations of fluid dynamics also give rise to more complex and interesting behaviors—boundary layers, wakes, instabilities, shock waves, and turbulence. We have studied these phenomena only in the very simplest contexts. But I hope that I hvae given you a foundation that will help you can approach the more intricate situations in which you will encounter these complex phenomena in your future work.