Jan. 21+

Low Reynolds Number

In the past two lectures, we have studied simple flows of viscous fluids. We defined the Reynolds number,

$$R = \frac{VL}{\nu}$$

which characterizes the importance of viscosity in a given flow situation. In the next lecture, we will begin our study of the transition from low to high Reynolds number. First, though, there are some things that we should understand about the regime of extremely small Reynolds number.

There are two motivations for this study, both of which come from applications of physics to biology. Many physicists are now interested in using advanced experimental methods to probe the working of biological systems at the the cell and even at the single-molecule level. In addition, physicist and materials scientists are able to fabricate microscopic structure easily and precisely. It is interesting from this point of view to think about chemistry and biochemistry in artificial structures of very small volume. Looking at the formula above, it is clear that very small size structure have very small values of the Reynolds number.

I will first discuss the motion of bacteria in water. There is a very beautiful and pedagogical reference for this subject: E. M. Purcell, Am. J. Phys. 45, 3 (1977).

An E. Coli bacterium has a size of about 1 μ. It moves at a typical velocity of 30 μ/sec. So the flow around a bacterium corresponds to a Reynolds number of

$$R = \frac{3\times10^{-3}\,\text{cm/sec}\cdot 10^{-4}\,\text{cm}}{10^{-2}\,\text{cm}^2/\text{sec}} = 3\times10^{-5}$$

For a person swimming with molasses (with $\nu \sim 10$ cm^2/sec), the same Reynolds number corresponds to a speed $v \sim 10^{-2}$ cm/sec or $v \sim$ cm/min. Because of its small size, the dynamics of an E. Coli is completely dominated by viscosity.

If we approximate the E. Coli as a sphere, the equation of motion for a coasting bug is

$$M \frac{dv}{dt} = - 6\pi\eta R v$$

From this formula, the bug comes to rest exponentially in a time

$$\tau = \frac{M}{6\pi\eta R} \sim \frac{R^2}{\nu} \sim \frac{10^{-8} cm^2}{10^{-2} cm^2/sec} \sim 1 \mu sec$$

In this time, it moves

$$v\tau \sim 10^{-8} cm \sim 1 \mathring{A}$$

Thus, there is no coasting and no inertia for an E. Coli. In its world, Aristotle's laws of motion apply.

This means that every movement of an E. Coli must be created by generating a force and using up energy. The applicable equation of motion is

$$0 = F - 6\pi\eta R \frac{dx}{dt}$$

so that

$$\frac{dx}{dt} = v = \frac{F}{6\pi\eta R}$$

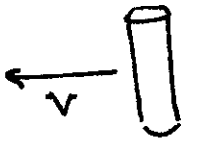
The force required to move the bug is about 0.1 pN (pico-Newton).

What the E. Coli has to provide this force is a whiplike appendage called a *flagellum.*

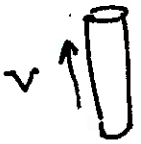
As the flagellum moves through water, the water exerts a friction force on the it that is translated back to the body of the bacterium. An idealization of this problem is that of a cylinder moving through water at low Reynolds number. By dimensional analysis, the force on a cylinder of length ℓ is given by

$$F/\ell \sim C\eta v$$

Oddly, the radius of the cylinder cannot appear as a factor. A more careful solution gives a logarithmic dependence on R/ℓ, depending on the direction of motion



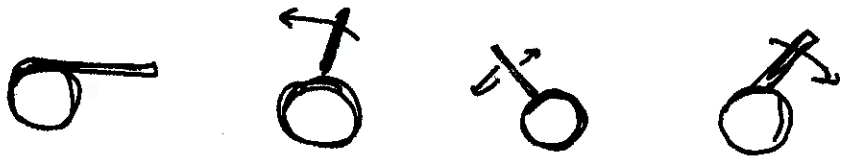
$$F/\ell \sim C_\perp \eta v \log \ell/R$$



$$F/\ell \sim C_\parallel \eta v \log \ell/R$$

Intuitively, $C_\parallel/C_\perp$ should be less than 1. For very long cylinders $C_\parallel/C_\perp = \frac{1}{2}$. For shorter cylinders, this ratio is larger, more typically $C_\parallel/C_\perp \sim \frac{2}{3}$.

One strategy for E. Coli to swim would be to deply the flagellum as a rigid oar.



But note that, over a cycle, $\oint dt\vec{F} = 0$. Then, from the equation of motion

$$\oint dx = 0$$

and the bug goes nowhere. What is needed, instead, is a strategy for which there is a net force over a cycle

$$\oint dt\, F \neq 0$$

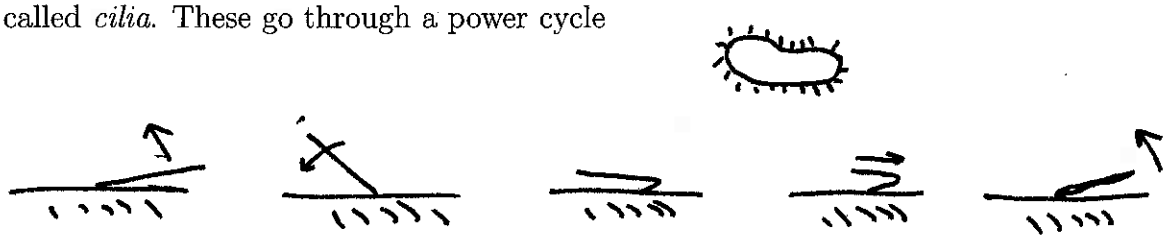
It is possible to make an analogy from this situation to motion on a manifold with nonzero Riemann curvature.

The strategy actually used by E. Coli is to rotate the flagellum, using the fact that the flagellum is curved with a definite handedness.

The E. Coli advances through water just as a bottle opener advances in a wine cork. This model was proposed by Berg and Anderson. It received a quite unusual

confirmation from work of Silverman and Simon, who created mutant E. Coli in which the flagellum was absent. Some of the bugs became attached to the walls of the container by the hook that attaches to the flagellum, and they were observed to spin around this attachment. In thye past 10 years, Berg, Steve Block, and other have studied the motor that spins the flagellum using optical tweezers. The motor generates a measurable torque of size about 3 pN-μ. This is the correct order of magnitude needed for propulsion.

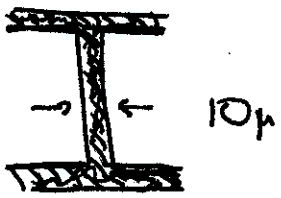
Paramecium uses a different type of propulsion stroke. Again, it must be such as to generate a new force around a cycle. Paramecia are covered with short segments called *cilia*. These go through a power cycle



that maues use of $C_{\|} < C_{\perp}$ in another way.

The main feature that simplifies the fluid dynamics of bacteria is their small size. It is possible to gain a similar simplification in technology by constructing very small channels for fluid flow. Then we can use the low Reynolds number flow to precisely control a chemical or biochemical reaction. This technology is called *microfluidics*. Stephen Quake at Stanford is a leading expert in this subject; I recommend his review: Quake and Squires, Rev. Mod. Phys. 77, 977 (2005).

To obtain a first notion of microfluidics, imagine that we put down a shape such as

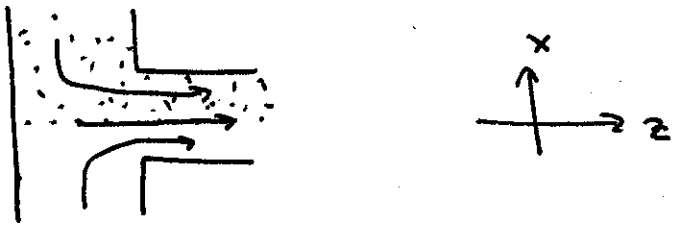


on a lithographic master. Then we can etch this pattern in plastic or use it to polymerize a gel. Either method would create a *microfluidic channel* with this size and shape.

The Reynolds number for a flow in this channel is

$$R \sim \frac{v \cdot 10^{-3}\,\text{cm}}{10^{-2}\,\text{cm}^2/\text{sec}} \sim \frac{v}{10\,\text{cm/sec}}$$

So, as long a v stays below m/sec, we are in the regime of simple laminar flow.



Mixing of reagents in the channel occurs by diffusion only. The diffusion of a reagent at concentration c is a fixed background flow is described by the equation

$$\frac{D}{Dt} c = \frac{\partial c}{\partial t} + \vec{v} \cdot \vec{\nabla} c = D \nabla^2 c$$

The simplest situation is a long channel in the $\hat{z}$ direction with $v = V_0 \hat{z}$. Then for steady flow

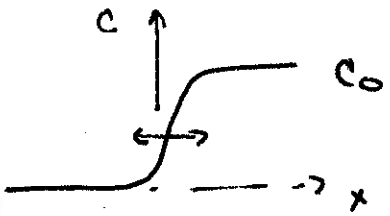
$$V_0 \frac{\partial}{\partial z} c = D \nabla^2 c$$

This is a diffusion equation with the variable z/V_0 playing the role of time. For a problem with the initial condition

$$c(x,z) = \begin{cases} c_0 & x > 0 \\ 0 & x < 0 \end{cases} \quad \text{at } z = 0$$

we find

$$c(x,z) = c_0\left[\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(x/\left[\frac{4Dz}{v_0}\right]^{\frac{1}{2}}\right)\right]$$



$$\Delta x \sim \left(\frac{4Dz}{v_0}\right)^{\frac{1}{2}}$$

We might describe this situation by a dimensionless number called the *P'eclet number*. If w is the width of the channel,

$$Pe = \frac{w^2}{D} \Big/ \frac{w}{v_0} = \#\text{ of widths of flow needed for mixing}$$

If D is the diffusion constant for the reagent.

$$Pe = \frac{v_0 w}{D} = R \cdot \frac{v}{D}$$

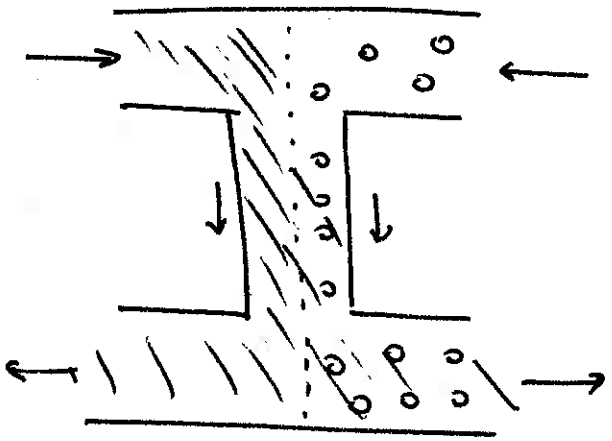
If the channel has a length greater than $w \cdot$ Pe, while $R < 1$, we have controlled mixing.

In biochemistry, D has a large variation among chemical species.

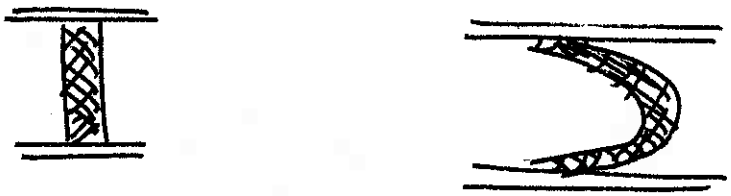
ions in solution	$D \sim 10\ cm^2/sec$
proteins	$D \sim 0.4\ cm^2/sec$
virus	$D \sim 0.02\ cm^2/sec$

so we can create a microfluidic filter in which

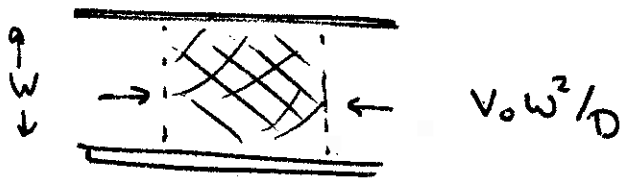
$\frac{l}{w} > Pe$ for some species $\frac{l}{w} < Pe$ for others



It is also possible to arrange a microfluidic channel in which the mixing is done by Poiseuille flow. This process is called *Taylor dispersion.* In a narrow channel, a layer of reagent will deform with flow



The reagents spread by a distance $v_0 t$, where v_0 is the fluid velocity in the center of the channel. After a diffusion time $\Delta t = w^2/D$, the situation blurs to



If we continue the flow, the material will be mixed further. After N steps, $\Delta t = Nw^2/D$, and we fill a volume of pipe of length $\sqrt{N} v_0 w^2/D$. The effective diffusion constant is

$$D_z = \frac{(\Delta z)^2}{\Delta t} = \frac{v_0^2 w^2}{D}$$

or

$$D_z = (Pe)^2 \cdot D$$

The article of Quake and Squires reviews many other effects that can be achieved in microfluidics by exploiting other properties of a fluid. Variable density, surface tension, polymer dynamics, and electrostatic forces can all be exploited to control chemical reactions in these small geometries.