Physics 211 – Final Exam

The exam is worth a total of 100 points, 50 points per problem.

The exam is open-book. However, if you make strong use of a reference other than the class textbooks and notes, please cite the reference in your solutions.

1. (25 points) Consider the laminar flow around a sphere of radius a at high Reynolds number. Let the fluid velocity at infinity be $V_0\hat{z}$. The flow pattern should be that of incompressible irrotational flow, except in a boundary layer and a wake region. However, it is not so easy to solve for the size of the boundary layer and for the drag force. The boundary layer is thick on the front side of the sphere but grows toward the back, and the transition to the wake region is complicated. In this question, I ask you to calculate the drag force on the sphere by making some approximations that are acceptable only on a take-home exam. In particular, I ask you to assume that the boundary layer has a thickness independent of position and that the incompressible irrotational solution is not modified outside of this boundary layer.

   (a) Assume that (i) the boundary layer has a fixed thickness equal to
   $$\Delta = (\nu a/V_0)^{1/2} ,$$
   and (ii) that the tangential component of the fluid velocity varies linearly from the surface of the sphere, where $v_\parallel$ must be zero, to the edge of the boundary layer. Compute the force on the sphere per unit area due to viscosity as a function of the position on the sphere under these assumptions.

   (b) Under the assumptions above, the pressure makes a negligible contribution to the drag force. Why?

   (c) Compute the drag force.

2. (25 points) Consider a small, very hot point probe in a flow $\vec{v} = V_0\hat{z}$. For simplicity, take the temperature at infinity to be zero. Take the thermal conductivity of the fluid to be χ.

   (a) Write the equation for the time-independent temperature distribution $T(\vec{x})$ in this flow.

   (b) The assumption that the flow is not modified by the probe is equivalent to the assumption that $\chi \gg \nu$. Explain this. (The thernal expansion of the fluid is not important in this explanation, and, indeed, you can ignore that effect.)

   (c) Show that the function
   $$T(\vec{x}) = \frac{\mathcal{T}}{|x|}$$
   solves the equation of part (a) when $V_0 = 0$. The probe then heats up a spherical region around it.

(d) For the case $V_0 > 0$, solve the equation of part (a) by Fourier transformation. You might need to know that the Fourier transform of $1/(k^2 + A^2)$ is a Yukawa potential

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + A^2} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi|x|} e^{-A|x|}$$

Normalize your solution so that it goes to the solution of part (b) as $|x| \to 0$.

(e) Sketch the distribution of temperature given by the solution of part (d). Is this physically correct?

3. (50 points) Consider a long cylinder of incompressible fluid held together by its own gravity. In this problem, you will analyze the small oscillations of this system. For simplicity, consider modes with no dependence on z, reducing this to a 2-dimensional problem. You may ignore the viscosity of the fluid.

(a) Analyze the unperturbed situation. Show that the surface gravity is

$$g = 2\pi G\rho R$$

where ρ is the density of the fluid, R is the radius of the cylinder, and G is Newton's constant.

(b) Find the pressure as a function of r inside the cylinder.

(c) The perturbed situation is defined by perturbations

$$\delta v_r, \quad \delta v_\phi, \quad \delta p, \quad \delta\Phi, \quad \delta\zeta$$

where Φ is the gravitational potential and $\delta\zeta$ – a function of ϕ – is the vertical displacement of the surface. It is convenient to combine δp and $\delta\Phi$ into

$$\delta W = \frac{\delta p}{\rho} + \delta\Phi$$

Linearize the Navier-Stokes equation, and write out the $\hat{r}$ and $\hat{\phi}$ components of this equation. Write the linearized form of the equation of continuity.

(d) Fourier transform in t and ϕ,

$$\delta v_r = v_r(r) e^{-i\omega t + im\phi}\ , \quad \text{etc.}$$

Show that the Fourier coefficients obey

$$-i\omega v_r = -\frac{d}{dr} W$$

and

$$-i\omega \frac{1}{r}\frac{d}{dr}(r v_r) = -m^2 \frac{W}{r^2}$$

(e) Elminate v_r and obtain an equation for $W(r)$. Show that the solution of this equation that is regular at the origin is

$$W(r) = Ar^m$$

(f) Away from the boundary near $r = R$, the perturbation $\delta\Phi$ obeys a *sourceless* Poisson equation. Show from this that the perturbation of the gravitational potential satisfies

$$\Phi(r) = Br^m \quad \text{inside} \qquad = Cr^{-m} \quad \text{outside}$$

(g) There are three more boundary conditions to match. $\vec{g} = -\vec{\nabla}\Phi(r)$ should be continuous at the surface, that is, at $r = R + \zeta$. The pressure p should be zero at the surface, and outside. The relation

$$v_r(R) = -i\omega\zeta$$

gives a needed fourth constraint among these quantities. Using these constraints, eliminate A, B, and C in terms of ζ.

(h) Find a formula for the oscillation frequency $\omega^2(m)$.

(i) Your result in (h) should be that $\omega^2 = 0$ for $m = 1$ and $\omega^2 < 0$ for $m = 0$. Explain these results physically.