Physics 211 – Final Exam

The exam is worth a total of 100 points.

The exam is open-book. However, if you make strong use of a reference other than the class textbooks and notes, please cite the reference in your solutions.

1. (40 points) A non-Newtonian fluid is one in which the stress due to shear is a nonlinear function of the shear. For example, if you mix cornstarch and water, the stress is large for small shear, but the fluid flows more easily at large shear. In this problem, assume that the fluid in question is incompressible, with fixed density ρ.

   (a) In a Newtonian fluid, we wrote

   $$T^{ij} = -2\eta\sigma^{ij} \quad \sigma^{ij} = \frac{1}{2}(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i})$$

   For a weakly non-Newtonian fluid, a 'sensible' formula is

   $$T^{ij} = -2\eta\sigma^{ij} + A\sigma^{ij}(\sigma^{kl})^2 + \cdots$$

   (The negative sign represents a decrease in stress under high shear.) Why should the term quadratic in σ^{ij} be absent?

   (b) For $\vec{v}$ parallel to $\hat{x}$ and a function of y only, write $v_{,y} = \partial v/\partial y$. Then a possible formula for T^{xy} is

   $$T^{xy} = -\eta\sigma_0 \sinh^{-1}(v_{,y}/\sigma_0)$$

   where σ_0 is a constant. Consider the flow of this fluid between parallel plates oriented normal to the $\hat{y}$ direction and separated by a distance a, if the fluid is forced with a constant pressure gradient

   $$G = -\frac{dp}{dx}$$

   as in Poiseuille flow. Work out the formula for the velocity profile in y. Find a formulae for the mass flow (g/sec per unit length in z).

   (c) For this non-Newtonian fluid, work out the corresponding formulae for Poiseuille flow through a pipe of radius A.

   (d) Show that, for large values of G, the fluid flow in both cases is characterized by thin boundary layer near the surface. What is the thickness of the boundary layer?

(e) For the problem of part (b), assume that the two plates have a temperature gradient

$$T(x) = T_0 + Bx$$

If the thermal diffusion constant of the fluid is κ (and we assume that the temperature current is linear in the gradient of T as usual), find the distribution of $T(x, y)$ between the plates. Ignore heating of the fluid by viscous stresses.

2. (60 points) Consider a slice of the atmosphere or the ocean. Choose coordinates z vertical, y north-south, and x east-west. Let the slice lie between two lines of latitude, $y = 0$ and $y = Y$. Make several idealizations: Include the Coriolis force, but ignore the variation of the Coriolis force with latitude, so $f = 2\Omega \cos\theta =$ constant. In fact, assume $\vec{f}$ parallel to $\hat{z}$. Treat the fluid in the Boussinesq approximation, with density $\rho = \rho_0(1-\alpha\Delta T)$. Consider the slice of fluid as having fixed height H, with the pressure set to a constant at the top. (In the atmosphere, this would be the *tropopause*, in the ocean, sea level.) For definiteness, put $z = 0$ at the top, and $z = -H$ at the bottom. Ignore the viscosity and the thermal diffusion constant.

(a) Let the fluid have a temperature gradient in y: $T(y) = T_0 + Ay$. Show that this leads to a pressure gradient, depending on y, that also increases linearly with depth. Show that there is a simple solution in which the fluid has a velocity in the $\hat{x}$ direction that increases linearly with depth. Find the formula for $\vec{v}(z)$.

(b) This configuration might be unstable. If a fluid velocity arises in the $\pm\hat{y}$ direction, the fluid convects T, affecting the distribution of pressure. Give a handwaving argument that there is an instability, and sketch the resulting flows and temperature gradients.

(c) Write the equations describing this system. The variables are v_x, v_y, v_z, p, T, so you need five equations. Write them, and linearize them about the state found in (a).

(d) A very simple approximation for treating these equations is to assume that the small deviations in v_x and v_y are independent of z. We can get equations for these two parameters as a function of x and y by integrating the equations from part (c) over z. Show that this implies, for

$$\Lambda = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \ ,$$

that $\Lambda = 0$. From this, show that the convection of T cancels out of the equations for v_x and v_y.

(e) Here is a better, but still very crude strategy. Divide the slice into two layers of equal thickness in z. The interesting action will be in the bottom layer. While $v_z = 0$ at $z = -H$, we can allow v_z to be nonzero at the top of the bottom layer: $v_z(-H/2) = V_z$. Then, if we model v_x, v_y as independent of z through this layer, show that $\Lambda H/2 + V_z = 0$. Now we can treat v_x, v_y, p, T as independent of z in the bottom layer and we get a closed set of equations. The linearized equations have no explicit dependence on y.

(f) Expand the linearized equations as:

$$v_x(x, y, t) = v_x \exp[-i\omega t + ik_x x + ik_y y]$$

and similarly for the other variables. [I am using periodic boundary conditions in y, which is illustrative, not realistic.] Then we get five algebraic equations. Write them out. Replace the v_z equation by the simple form

$$0 = g\rho\alpha T(H/2) + p \ .$$

(which is an approximation to $g\rho\alpha T - dp/dz = \cdots$).

(g) Solve these equations. A convenient way to do this is to find equations for Λ in part (d) and for

$$\Omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \ ,$$

the vorticity in the bottom layer. Find an equation for the dispersion relation for ω.

(h) The equation for ω becomes simpler in the limit $|k_y| \gg |k_x|$. Starting from this limit, discuss the stability of the flow pattern.