Physics 211 – Final Exam

The exam is worth a total of 100 points, 50 points per problem.

The exam is open-book. However, if you make strong use of a reference other than the class textbooks and notes, please cite the reference in your solutions.

1. (50 points) This problem concerns the flow of water down a drain.

   (a) Consider a large tank of fluid with a horizontal floor that contains a small circular hole of radius a. Make the approximation that the vertical component of the fluid velocity at the hole is constant across the hole and equal to $\mathcal{V}$. Assuming potential flow, find $\vec{v}(\vec{x})$ for a flow of the fluid into the hole. You can do this by setting up an analogue electrostatics problem and solving it. You may assume that $|\vec{x}| \gg a$.

   (b) Now add a swirl to the flow, that is, add a fluid velocity in the $\hat{\phi}$ direction and assume it is much larger than the flow in $\hat{r}, \hat{z}$. If we ignore viscosity, the angular momentum of a fluid element will be conserved. Why? Assuming that $v_\phi = V_0$ at $r = a$, compute v_ϕ as a function of r. Compute the vorticity.

   (c) If the fluid has a finite height above the drain, the surface will dip as $r \to 0$. Compute the shape of the surface for the flow described in part (b).

   (d) In a rotating coordinate system, the flow in part (a) will develop a small swirl. Why? In which direction, in the northern hemisphere of the Earth? (Caution: This is much too small an effect to observe reliably in your bathtub.)

   (e) Now put a cylindrical wall at $r = a$. A boundary layer will form at this wall, where the fluid's viscosity will act to slow down the flow. Start from a situation in which, after stirring, $v_\phi = V_0$ near $r = a$ and goes very rapidly to zero at the wall. I claim that, if the stirring is stopped, a boundary layer will grow out of the wall with time. To analyze this claim, write the $\hat{\phi}$ component of the Navier-Stokes equation, assuming an axisymmetric flow with only $v_\phi(r, t)$ not equal to zero. Make approximations appropriate to a boundary layer of thickness $\delta \ll a$. Solve the equation, and determine the thickness of the boundary layer as a function of time.

2. (50 points) In this problem, water is placed in a pipe of radius a bent into a rectangle. The rectangle is oriented in the $\hat{x}, \hat{z}$ plane ($\hat{z}$ vertical) with width w and height h. The bottom is heated and the top is cooled.

   (a) Taking account of the viscosity of water, the water flows around the pipe in a Poiseuille flow. Let V be the mean velocity of this flow, so that the mass/sec passing a given point on the tube is $\rho \pi a^2 V$. Find the velocity distribution in across the tube as a function of radius. Assume that the fluid is incompressible;

then V is a constant around the circle. Compute the force that the pipe exerts on the fluid, and write an equation for the damping of a flow around the pipe in the form

$$\frac{dV}{dt} = -AV$$

What is A? You may ignore the effect of the corners of the rectangle.

(b) Now assume that the temperature of the fluid can vary around the pipe. For simplicity, assume that the temperature is constant across the cross section of the pipe and convects at a velocity equal to the velocity at the center of the pipe. (What is the relation between this velocity and V?) Assume that the vertical pipes are insulated and that the horizontal pipes are good conductors of heat, so that so that $T = T_2$ along the bottom pipe and $T = T_1$ along the top pipe. Assume that $\Delta T = (T_2 - T_1) > 0$. Write an equation for the temperature in the vertical pipes $T(t, z)$, including convection and diffusion. For fixed, time-independent V, solve for the equilibrium distribution $T(z)$ in each of the vertical pipes.

(c) Assume that the fluid is a good enough conductor that this equilibrium is re-established rapidly as V changes in time. Using the Boussinesq approximation, work out the bouyancy force on the fluid in the pipe as a function of V. Expand the result for small V, and show that this adds a term of the form

$$+BV + \cdots$$

on the right-hand side of the equation in part (a). What is B?

(d) Show that, when ΔT is sufficiently small, the fluid is stable at $V = 0$. Find the threshold for an instability.

(e) Write the equation for dV/dt for values of V that are not necessarily small, assuming that the assumption of rapid equilibration used in part (c) remains valid. Above the threshold computed in part (d), what is the behavior of $V(t)$? A qualitative or graphical solution is sufficient.