Feb. 23

Compressible Fluids

In the previous lecture, in our discussion of heating and convection, we also generalized from incompressible fluids to fluids with a small compressibility and coefficient of thermal expansion. However, there are important examples of fluids—especially, gases—which undergo large changes in volume under pressure. In this lecture, I would like to illustrate some new effects that result from these large compressibilities. The most important of these effects is the appearance of compressibility waves—sound waves. We will also discuss new instabilities that take advantage of compressibility.

Compressibility also can lead to a new sort of pathology in the partial differential equations describing the fluid, leading to singular behavior of the solutions: shock waves. I will postpone this aspect of the subject to a later lecture.

To begin, I will write the basic equations of a compressible fluid and derive the equation of sound waves. Allowing variable density but ignoring viscosity and heat transfer, these basic equations are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p$$

Starting from an equilibrium state with fixed ρ and p and $\vec{v} = 0$, expand to linear order in fluctuations about this state:

$$\frac{\partial}{\partial t}\delta\rho + \rho\,\vec{\nabla}\cdot\delta\vec{v} = 0$$

$$\frac{\partial}{\partial t}\delta\vec{v} + \frac{1}{\rho}\vec{\nabla}\delta p = 0$$

These are 4 equations in 5 variables. To close the set of equations, we need to relate p to ρ. The relation between pressure and density is the *equation of state* of the fluid, which is derived from its thermodynamic properties. This relation also depends on the time scale of the fluctuations. The simplest assumption is that the compressions of a gas involved in sound waves are sufficiently slow that the gaseous medium remains at a constant temperature. Newton already realized that this assumption of *isothermal* compression does not give the correct speed of sound. A better assumption is that the fluctuations take place on too rapid a timescale for substantial heat flow. Then the fluctuations are *adiabatic*, that is, taking place at constant *entropy* per gram of fluid. In this case, we relate pressure to density variations by

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho$$

Now we have a closed system of equations. These can be simplified by taking the time derivative of the mass conservation equation,

$$\frac{\partial^2}{\partial t^2}\delta\rho = -\rho\vec{\nabla}\frac{\partial \vec{v}}{\partial t} = \rho\,\vec{\nabla}\cdot\vec{\nabla}\,\delta\rho\cdot\frac{\left(\frac{\partial p}{\partial \rho}\right)_s}{\rho}$$

Then we find

$$\frac{\partial^2}{\partial t^2}\delta\rho = c^2\,\nabla^2\delta\rho$$

where

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

We identify c with the speed of sound. For example, the general solution of this wave equation for 1-dimensional disturbances is

$$\delta\rho = f(x-ct) + g(x+ct)$$

and this makes it obvious that the perturbations propagate with the velocity c.

In the dynamics of gases, it is typically a good approximation to use the ideal gas law as the equation of state. I will now review the basic formulae that follow from that assumption.

In the notation of this course, the ideal gas law is written

$$pV = \frac{k_B T}{m}$$

where $V = 1/\rho$, the volume per gram, k_B is Boltzmann's constant, and m is the mass of a gas molecule. For an isothermal compression,

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{k_B T}{m} = \frac{p}{\rho}$$

But, this is not correct for an adiabatic compression. In that case, the gas heats up due to the work done on it, and this heat is not released to the environment. Instead of the equation above, standard thermodyamic arguments give

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \gamma \frac{p}{\rho}$$

where $\gamma > 1$ is a constant associated with the properties of the gas. Then the law of adiabatic compression or expansion is

$$p \sim \rho^{\gamma} \qquad \text{or} \qquad p \sim V^{-\gamma}$$

and the speed of sound is given by

$$c = \frac{\gamma p}{\rho}$$

For an ideal gas, the internal energy and enthalpy are given by

$$u = c_V \frac{k_B T}{m} \qquad h = c_P \frac{k_B T}{m}$$

where

$$c_P = c_V + 1 \qquad \text{and} \qquad \gamma = \frac{c_P}{c_V}$$

For a monatomic gas,

$$c_V = \frac{3}{2} \qquad c_P = \frac{5}{2}$$

for a diatomic gas

$$c_V = \frac{5}{2} \qquad\qquad c_p = \frac{7}{2}$$

and, more generally, the value of c_v reflects the number of internal degrees of freedom of the molecules of which the gas is made. Then

$$c_V = \frac{1}{\gamma - 1}$$

and

$$c_p = \frac{\gamma}{\gamma - 1}$$

Thus,

$$\gamma = \begin{cases} 5/3 & \text{monatomic gas} \\ 7/5 & \text{diatomic gas} \end{cases} \qquad \text{etc.}$$

These statements will be proved in your statistical mechanics course.

I will now work some fluid dynamics examples that make use of these concepts. For the first example, I will consider waves in a compressible atmosphere. This theory is applicable to perturbations of stellar structure. Recently, for example, we have learned much about the structure of the sun by *helioseimology*, the measurement of the spectrum of oscillations visible on the sun's surface. For a detailed theory of helioseismology, one needs to consider the spherical geometry of the sun and its nonzero rotation. Here, I will ignore both of these factors and analyze a simple planar, vertically stratified atmosphere.

Before we consider any fluctuations, the atmosphere is in hydrostatic equilibrium

$$\frac{dp_0}{dz} = -\rho_0 g$$

where now $p_0(z)$, $\rho_0(z)$ represents a nontrivial distribution of pressure and density determined by the equation of state. At the height z, the speed of sound is

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p_0(z)}{\rho_0(z)}$$

The basic equations governing this system are: conservation of mass

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

conservation of momentum

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - g\hat{z}$$

and conservation of entropy

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{p}{\rho^\gamma}\right) = 0$$

Now expand these equations to first order about the equilibrium

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\delta\vec{v} + \delta\vec{v}\cdot\vec{\nabla}\rho_0 = 0$$

$$\frac{\partial}{\partial t} v_{x,y} = -\frac{1}{\rho_0}\nabla_{x,y}\delta p$$

$$\frac{\partial}{\partial t} v_z = \frac{\delta\rho}{\rho_0^2}\frac{dp_0}{dz} - \frac{1}{\rho_0}\frac{d}{dz}\delta p$$

$$\frac{1}{\rho_0^\gamma}\frac{\partial}{\partial t}\delta p - \frac{\gamma p_0}{\rho_0^{\gamma+1}}\frac{\partial}{\partial t}\delta\rho + \delta v_z \frac{d}{dz}\left(\frac{p_0}{\rho_0^\gamma}\right) = 0$$

The problem is translation invariant with respect to x, y, t, so introduce Fourier components in these directions

$$\delta\rho = \rho(z)\, e^{-i\omega t}\, e^{i\vec{k}_\perp\cdot(x,y)} \qquad \text{etc.}$$

The equations for the individual Fourier modes are

$$-i\omega\rho + \rho_0\left(i\vec{k}_\perp\cdot\vec{v}_\perp + \frac{dv_z}{dz}\right) + v_z\frac{d}{dz}\rho_0 = 0$$

$$-i\omega v_{x,y} = -ik_{x,y}\frac{p}{\rho_0}$$

$$-i\omega v_z = \frac{\rho}{\rho_0^2}\frac{dp_0}{dz} - \frac{1}{\rho_0}\frac{dp}{dz}$$

$$-i\omega\left(\frac{p}{\rho_0} - c^2\frac{\rho}{\rho_0}\right) + v_z\rho_0^{\gamma-1}\frac{d}{dz}\left(\frac{p_0}{\rho_0^\gamma}\right) = 0$$

The equations for v_x, v_y are very simple

$$v_{x,y} = \frac{k_{x,y}}{\omega}\frac{p}{\rho_0}$$

and so we can eliminate these variables. They appear only in the mass equation, which now reads

$$-i\omega \frac{\rho}{\rho_0} + \frac{dv_z}{dz} + i\,\frac{k_\perp^2}{\omega}\frac{p}{\rho_0} + \frac{v_z}{\rho_0}\frac{d\rho_0}{dz} = 0$$

Introduce

$$\delta h = \frac{\delta p}{\rho_0}$$

This is the perturbation of enthalpy, since

$$dh = \frac{dp}{\rho} + T\,ds = \frac{dp}{\rho} \quad \text{if } ds = 0$$

Now notice that derivatives of $p(z)$ and $v_z(z)$ appear in these equations, but ρ/ρ_0 appears only algebraically, so we can eliminate it. The entropy equation reads

$$\frac{\rho}{\rho_0} = \frac{1}{c^2}h - i\,\frac{v_z}{\omega}\,\frac{\rho_0^\gamma}{\gamma p_0}\,\frac{d}{dz}\left(\frac{p_0}{\rho_0^\gamma}\right)$$

Let

$$A = -\frac{\rho_0^\gamma}{\gamma p_0}\,\frac{d}{dz}\left(\frac{p_0}{\rho_0^\gamma}\right)$$

Then

$$\frac{\rho}{\rho_0} = \frac{1}{c^2} h + i \frac{v_z}{\omega} A \qquad\qquad \frac{1}{c^2} = \frac{\rho_0}{\gamma p_0}$$

We can use this equation to substitute for ρ/ρ_0 in the other relations. Substituting into the mass equation, we find

$$\frac{dv_z}{dz} + i \frac{k_\perp^2}{\omega} \frac{p}{\rho_0} + \frac{v_z}{\rho_0} \frac{d\rho_0}{dz} - i\omega \frac{1}{c^2} h$$

$$+ v_z \frac{\rho_0^\gamma}{\gamma p_0} \frac{d}{dz}\left(\frac{p_0}{\rho_0^\gamma}\right) = 0$$

so that

$$\frac{dv_z}{dz} + i \frac{k_\perp^2}{\omega} h - i \frac{\omega}{c^2} h + \cancel{\frac{v_z}{\rho_0} \frac{d\rho_0}{dz}} + \frac{v_z}{\rho_0} \frac{1}{c^2} \frac{dp_0}{dz} - \cancel{v_z \frac{1}{\rho_0} \frac{d\rho_0}{dz}} = 0$$

Using

$$\frac{1}{\rho_0} \frac{dp_0}{dz} = -g$$

we find, finally

$$\frac{dv_z}{dz} - \frac{g}{c^2} v_z + i\left(\frac{k_\perp^2}{\omega} - \frac{\omega}{c^2}\right) h = 0$$

Substituting into the z momentum equation, we find

$$-i\omega v_z = -g\left(\frac{\rho_0}{\gamma p_0} h + i\frac{v_z}{\omega} A\right)$$
$$- \left(\frac{d}{dz} h + \frac{1}{\rho_0} h \frac{d\rho_0}{dz}\right)$$

or

$$-i\omega v_z = -g\,\cancel{\frac{\rho_0}{\gamma p_0} h} - i\,\frac{g v_z}{\omega} A - \frac{dh}{dz}$$
$$+ \frac{\rho_0^{\gamma}}{\gamma p_0} h \frac{d}{dz}\left(\frac{p_0}{\rho_0^{\gamma}}\right) - \cancel{\frac{\rho_0}{\gamma p_0} h \cdot \frac{1}{\rho_0} \frac{dp_0}{dz}}$$

so that

$$\frac{dh}{dz} + A h = i v_z \left(\omega + \frac{A g}{\omega}\right)$$

We can replace $v_z(z)$ with the time derivative of $\xi(z)$, the vertical displacement of a fluid element from equilibrium. In Fourier space

$$v_z = -i\omega\,\xi$$

Then the two remaining equations become

$$\frac{d\xi}{dz} - \frac{g}{c^2}\xi + \frac{1}{c^2}\left(\frac{k_\perp^2 c^2}{\omega^2} - 1\right) h = 0$$

$$\frac{dh}{dz} - \frac{N^2}{g} h - (N^2 - \omega^2)\,\xi = 0$$

where

$$N^2 = -Ag$$

This is a well-defined system of equations that we can solve for the fluctuations about an explicit density profile. To get an idea of the content of these equations, though, I will make a further approximation, that the wavelength in the z direction is much shorter than the characteristic length on which the density profile varies in z. In this limit, we can approximate

$$\xi(z) \approx \xi\, e^{i k_z z}$$

and rewrite the equations as

$$i k_z \xi - \frac{1}{c^2}\left(1 - \frac{k_\perp^2 c^2}{\omega^2}\right) h = 0$$

$$i k_z h - (N^2 - \omega^2)\,\xi = 0$$

The two equations can be combined into

$$k_z^2 + \frac{1}{c^2}(N^2 - \omega^2)\left(1 - \frac{k_\perp^2 c^2}{\omega^2}\right) = 0$$

or

$$\omega^4 - (N^2 + k^2c^2)\,\omega^2 + N^2 k_\perp^2 c^2 = 0$$

This is a fourth-order equation in ω. Essentially, there are two distinct scalar waves that are coupled by the dynamics.

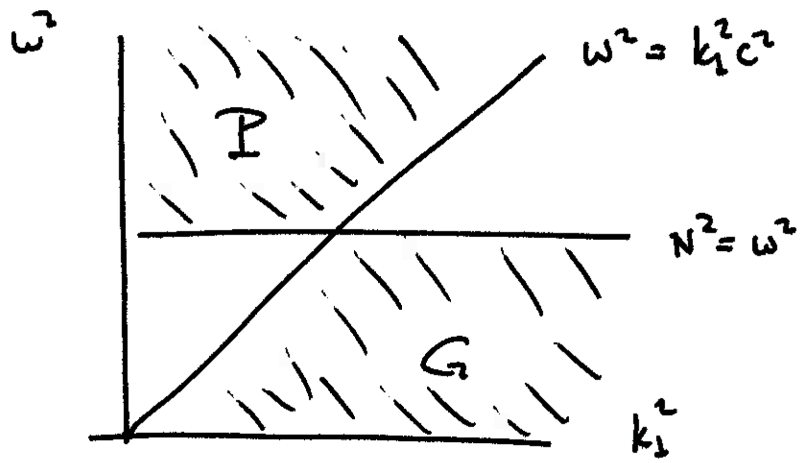
One way to pick apart the two components is to look at the condition for $k_z = 0$, that is, propagation in the x, y directions only. From the equation above, that condition is

$$(N^2 - \omega^2) = 0 \qquad \underline{\text{or}} \qquad \omega^2 = k_\perp^2 c^2$$

Modes that also propagate in z must have $k_z^2 > 0$, that is

$$(N^2 - \omega^2) < 0 \qquad \underline{\text{or}} \qquad (\omega^2 - k_\perp^2 c^2) < 0$$

but not both. We can graph these regions in the $(k_\perp^2, \omega^2)$ plane.



The region labelled P contains the modes that are acoustic models, or *p-modes.* These modes are dominantly composed of local oscillations in density, like sound waves. This region connects to $\omega^2 \to \infty$; in this limit, the dispersion relation becomes

$$\omega^2 \approx k^2 c^2$$

The region labelled G contains the modes that are gravity modes or *g-modes.* These modes are dominantly composed of local oscillations of the vertical density gradient, with gravity as the restoring force. This region connects to $\omega^2 \to 0$; in that limit, the dispersion relation becomes

$$\omega^2 = N^2 \frac{k_\perp^2}{k^2}$$

For the gravity modes, there is a strange relation between the phase velocity and the group velocity of the waves. The phase velocity, which determines the motion of the crests, is

$$\vec{v}_p = \frac{\omega}{k} \hat{k}$$

and points in the direction of $\hat{k}$. The group velocity, which determines the motion of wavepackets, is and points in the direction of $\hat{k}$. The group velocity, which determines the motion of wavepackets, is

$$\vec{v}_g = \frac{\partial}{\partial \vec{k}} \omega = - \frac{\omega k_z}{k_\perp^2} \left(\hat{z} - \hat{k}\, \hat{k} \cdot \hat{z} \right)$$

Thus, the group velocity is orthogonal to the phase velocity, and so energy propagates along the crests of the wave rather than perpendicular, as we have in the usual case. We will see more examples of such peculiar dispersion relations later in the course.

The second example that we can study is the collapse of a cloud of gas under gravity. I will analyze this first for a uniform cloud in deep space, for which the only forces of gravity come from the self-gravitation of the cloud.

The basic equations describing this situation are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \Phi$$

and the Poisson equation for the gravitational potential

$$\nabla^2 \Phi = 4\pi G \rho$$

The equilibrium state is one of uniform density ρ_0 and pressure p_0. Expanding to first order about this state

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\delta\vec{v} = 0$$

$$\frac{\partial}{\partial t}\delta\vec{v} = -\frac{1}{\rho_0}\vec{\nabla}\delta p - \vec{\nabla}\delta\Phi$$

$$\nabla^2 \delta\Phi = 4\pi G\delta\rho$$

I will relate the pressure fluctuations to the density fluctuations by

$$\delta p = c^2 \delta\rho$$

The unperturbed system is uniform in x, y, z, t, so we can Fourier transform in all four variables. The above equations then become

$$-i\omega \frac{\rho}{\rho_0} + i\vec{k}\cdot\vec{v} = 0$$

$$-i\omega\vec{v} = -i\vec{k}c^2\frac{\rho}{\rho_0} - i\vec{k}\Phi$$

$$-k^2\Phi = 4\pi G\rho$$

We can eliminate $\vec{v}$ by dotting the second equation with $\vec{k}$ and plugging the result back into the mass conservation equation

$$-\omega^2\frac{\rho}{\rho_0} + i\left(-ik^2c^2\frac{\rho}{\rho_0} - ik^2\Phi\right) = 0$$

Then, eliminating Φ in favor of ρ, we find

$$-\omega^2 \rho + k^2c^2 \rho - 4\pi G\rho_0 \rho = 0$$

which gives the dispersion relation

$$\omega^2 = k^2c^2 - 4\pi G\rho_0$$

The consequences of this equation are straightforward to see. For sufficiently large k, the fluctuations are sound waves shifted to lower frequency by the attractive gravitational interaction. However, for

$$k^2 < \frac{4\pi G\rho_0}{c^2}$$

there is an instability to gravitational collapse. This is the *Jeans instability.* The corresponding size of gravitational clumps is the *Jeans length*, given by

$$\lambda_J = \frac{\pi c^2}{G\rho_0}$$

The mass of the clumps is the *Jeans mass*,

$$M = \rho\lambda_J^3 = \left(\frac{\pi}{G}\right)^{3/2} \frac{c^3}{\rho_0^{1/2}}$$

The Jeans instability is the basic mechanism for forming stars from the bulk gaseous material of galaxies.

There is a useful physical interpretation of the parameters of the Jean length and mass. From the dimensionful quantities that play a role in this analysis, we can form two characteristic times. The first is the time for direct gravitational collapse

$$\tau_G = \left(\frac{\pi}{G\rho}\right)^{\frac{1}{2}}$$

The second is the time for sound signals to propagate across a gas cloud of size λ. For a sufficiently small cloud, the gas pressure is effective at preventing gravitational collapse. However, for a sufficiently large cloud, sound signals cannot propagate fast enough across the cloud to allow the gas pressure to stop the collapse. The criterion

$$\tau_s = \frac{\lambda}{c} > \tau_G$$

gives just the Jeans length.

As a second example of the Jean instability, I will now study the gravitational collapse of a planar layer of gas. This problem will have nontrivial dependence on z, the direction normal to the plane, but otherwise I will continue with the simplifying assumptions used in the previous example. I will consider the gas in isolation from any other gravitational sources, and I will use the simple relation

$$\delta p = c^2 \delta\rho$$

with constant c as the equation of state that relates pressure and density.

To set up this problem, we must first determine the equilibrium configuration that corresponds to hydrostatic equilibrium. The basic equations for equilibrium are

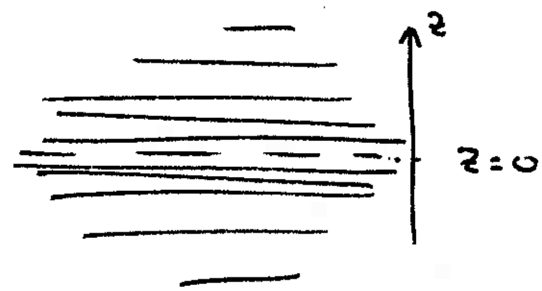
$$\frac{1}{\rho}\frac{dp}{dz} + \frac{d\Phi}{dz} = 0$$

and the Poisson equation

$$\frac{d^2}{dz^2}\Phi = 4\pi G \rho$$

I will assume that the equilibrium structure is up/down symmetric, that is,

$$\rho(z) = \rho(-z)$$



It is useful to let

$$m = \int_0^z dz\, \rho(z) \qquad \text{or} \qquad dm = dz\, \rho(z)$$

$$M = \int_0^\infty dz\, \rho(z)$$

Then M is half of the column height of mass (with units of g/cm^2). Using this definition and the equation of state, the equation of hydrostatic equilibrium becomes

$$c^2 \frac{d\rho}{dm} = -\frac{d\Phi}{dz}$$

$$c^2 \frac{d^2\rho}{dm^2} = -\frac{1}{\rho}\frac{d^2\Phi}{dz^2} = -4\pi G$$

and, combining this with the Poisson equation, we have

$$\frac{d^2\rho}{dm^2} = -\frac{4\pi G}{c^2}$$

This equation integrates to a quadratic dependence on m,

$$\rho = \frac{2\pi G M^2}{c^2}\left(1 - \frac{m^2}{M^2}\right) \qquad \text{so} \quad \rho(m=M) = 0$$

Now integrate the equation for dm/dz

$$\frac{dm}{dz} = \rho(m)$$

to find

$$M \tanh^{-1}\left(\frac{m}{M}\right) = \frac{2\pi G M^2}{c^2} z$$

Then the equilibrium density profile is

$$\frac{m}{M} = \tanh\left(\frac{2\pi G M}{c^2} z\right)$$

$$\rho(z) = \frac{2\pi G M^2}{c^2} \operatorname{sech}^2\left(\frac{z}{H}\right)$$

where the scale height H is

$$H = \frac{2\pi G M}{c^2}$$

Now we are ready to analyze the stability of this situation. The momentum equation is

$$\frac{\partial^2}{\partial t^2}\vec{\xi} = -\frac{1}{\rho_0}\vec{\nabla}\delta p + \frac{\delta\rho}{\rho_0^2}\vec{\nabla}p_0 - \vec{\nabla}\delta\Phi$$

where I have replaced the fluctuation of velocity by $\partial\vec{\xi}/\partial t$, where $\vec{\xi}$ is the displacement of a fluid element from equilibrium. Fourier transforming in time, we have

$$-\omega^2\xi = -\frac{1}{\rho_0}\vec{\nabla}p + \frac{\rho}{\rho_0^2}\vec{\nabla}p_0 - \vec{\nabla}\Phi$$

$$= -\vec{\nabla}W - \vec{\nabla}\Phi$$

where

$$\delta W = \frac{\delta p}{\rho_0} \quad \text{or} \quad W = \frac{p}{\rho_0}$$

This equation allows us to find the stability boundary: Stable oscillations correspond to $\omega^2 > 0$. A mode can go unstable by moving from positive to negative values of ω^2, so the boundary must be at $\omega^2 = 0$. The condition $\omega^2 = 0$ implies

$$W = -\Phi$$

Combining this with the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho$$

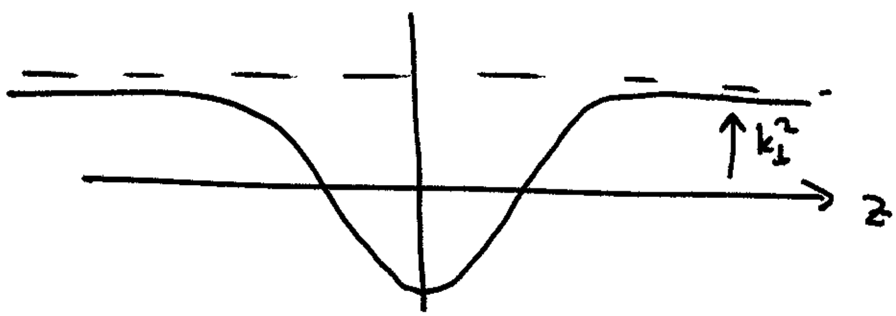
we find the relation

$$\nabla^2 W + \frac{4\pi G \rho_0}{c^2} W$$

Now introduce Fourier components in (x, y). The above partial differential equation becomes the ordinary differential equation

$$-\frac{d^2}{dz^2} W + \left(k_\perp^2 - \frac{4\pi G \rho_0}{c^2}\right) W = 0$$

which is essentially the Schrödinger equation with a potential of the form



The $1/\cosh^2(z/H)$ potential is a standard example in quantum mechanics. This potential has a bound state solution, corresponding to a negative energy for the case in which the potential goes to zero as $z \to \infty$. The bound state is at zero energy, as required for $\omega^2 = 0$, at the critical (Jeans) wavenumber

$$k_J^2 = \frac{2\pi G M}{c^2}$$

Density waves of longer wavelength, $k < k_J$ or $\lambda > \lambda_J$, are unstable to gravitational collapse.