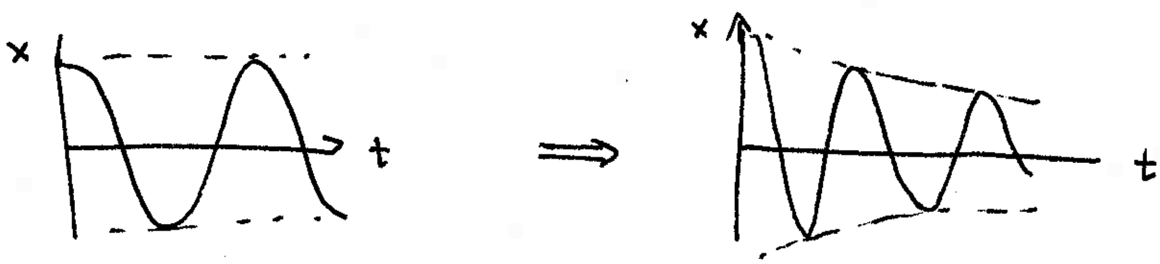
Jan. 26

Boundary Layers

So far, our study of viscous forces has dealt only with simple and smooth flows. In the next few lectures, I would like to study the complicating and even pathological properties of flows that arise where go turn up the fluid velocity from low to high Reynolds number.

In this lecture, I will discuss the concept of a *boundary layer.* This is a thin region with distinct behavior found in many situation of high Reynolds number. Mathematically, the theory of boundary layers is a type of *singular perturbation theory.* Let me begin by defining that concept and giving a simple example.

I assume you are familiar with *regular perturbation theory.* For example, consider adding a small damping term to a harmonic oscillator. Locally in time, there is only a small effect on the motion.



Some effects may accumulate over many cycles. For example, the energy slowly decreases and the phase slowly drifts relative to the zeroth-order solution. These are called *secular effects.* But, in any local region, the perturbed solution is close to the zeroth-order solution.

In contrast to this, we have the equation for the motion of an E. Coli discussed in the previous lecture,

$$M \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} = F$$

In a typical motion, the kinetic term is small, and so the *first* term can be treated as a small perturbation. For an initial condition $x(0) = x_0$, we can solve

$$\gamma \frac{dx}{dt} = F$$

and find

$$v(t) = \frac{F}{\gamma} \qquad\qquad x(t) = x_0 + \frac{F}{\gamma} t$$

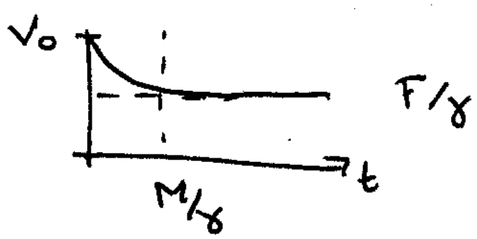
However, the full problem is a *second-order* differential equation that requires *two* independent boundary conditions (x_0, v_0). It is easy enough in this case to solve the full equation in the general case. The equation for $v(t)$ is

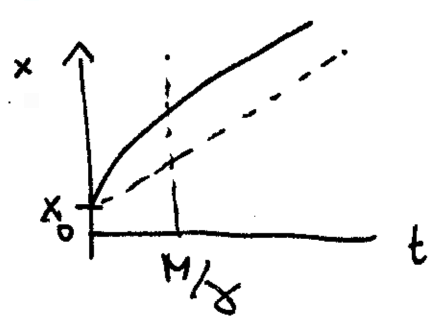
$$M \frac{dv}{dt} + \gamma v = F$$

and the solution is

$$v = \frac{F}{\gamma} + \left(v_0 - \frac{F}{\gamma}\right) e^{-\frac{\gamma}{M} t}$$

If $v_0 \neq F/\gamma$, then in a very short time $\tau = M/\gamma$ the velocity v damps to F/γ; after this,we go to the zeroth order solution written above.





Notice that the perturbation has an effect of order 1 in the small time interval $0 < t < M/\gamma$. To compute the effect of the perturbation, we need to blow up this region and solve carefully for the motion on this short time scale, then feed that information back to the solution of the problem on longer time scales.

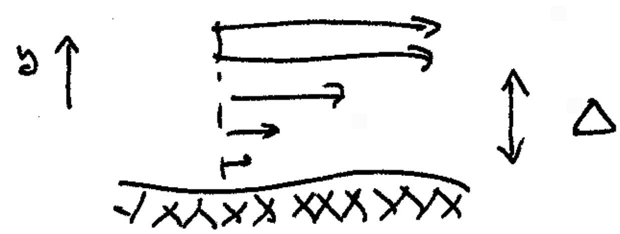
A *singular perturbation*, then, is one in which the perturbation contains *higher derivatives* than that zeroth-order equation. Typically, the singular perturbation has order 1 importance in a small region and we must account for this in assessing its effects.

Let's apply this insight to the Navier-Stokes equation at high Reynolds number.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v}$$

At first sight, it seems that, at large $\vec{v}$ or small ν, we can ignore the viscosity term. However, this too quick. If we remove the viscosity term, we obtain an equation that is first order in derivatives. We can impose the boundary condition $\hat{n}\cdot\vec{v} = 0$, but we cannot impose the full boundary condition $\vec{v} = 0$.

This seems to be a paradox. *Prandtl* resolved the paradox with the following idea: The parallel components of velocity go to zero in a very small region near the boundary.



If this region has thickness Δ, then in this region

$$\frac{dv}{dy} \sim \frac{1}{\Delta} \qquad \frac{d^2v}{dy^2} \sim \frac{1}{\Delta^2}$$

The viscosity term can have order 1 importance in this region if Δ is small enough that

$$\nu \cdot \frac{1}{\Delta^2} = \text{order } 1$$

or, in dimensionless terms,

$$\frac{1}{R} \frac{1}{(\Delta/L)^2} \cong 1$$

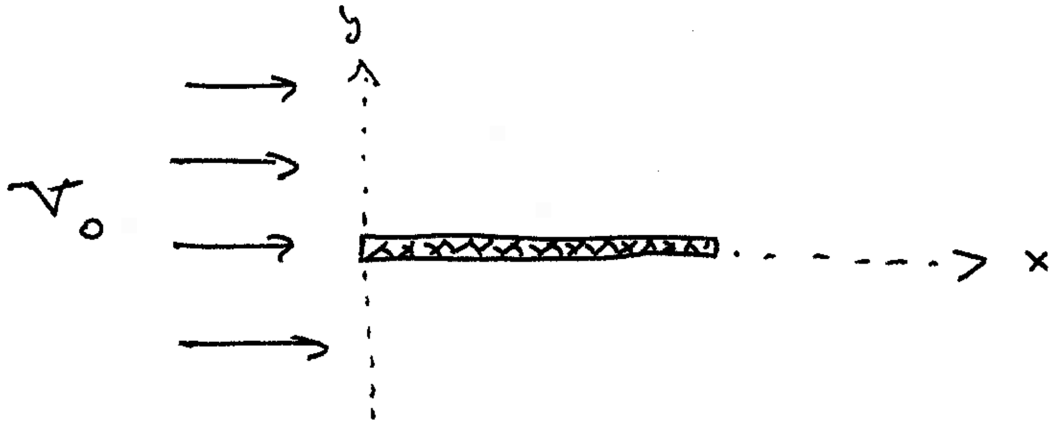
This small region is called a *boundary layer.* For large R and a problem with characteristic size L, the thickness of the boundary layer will be

$$\Delta = \frac{L}{\sqrt{R}}$$

As $R \to \infty$, the boundary layer becomes thinner and thinner, but it cannot be ignored. It is the fluid motion in this region that generates the tangential force of the fluid on the boundary surface, which is often what we want to solve for. This force often turns out to be larger than what we might naively expect.

To study the structure of a boundary layer, it is often useful to adopt a coordinate system in which we have blown up the thickness of the boundary layer to order 1 in the new coordinates. Often it is useful to carry out perturbation theory in the boundary layer using this coordinate system, carry out perturbation theory outside the boundary layer in the standard coordinate system, and then match the results. This is called the method of *matched asymptotic expansions.* I will put an example of this matching in the next problem set. However, the situation that I will study in this lecture does not require such a sophisticated analysis.

I would like now to analyze a specific, simple case of a boundary layer. Consider a thin plate of length L that is inserted into a smooth parallel fluid flow.



for which

$$\mathcal{R} = \frac{V_0 L}{\nu} \gg 1$$

Some interesting questions are: What is the shape of the region of fluid that is disturbed by the presence of the plate? What is the force exerted by the fluid on the plate?

I will first develop the problem formally. I will assume that the plate extends very far in the z direction, so that the problem is one of 2-dimensional flow. I will assume that the fluid is incompressible. I will look for a steady, time-independent solution. Then we must solve

$$(\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \nu \nabla^2 \vec{v}$$

Let $\vec{\omega} = \vec{\nabla} \times \vec{v}$. for a 2-dimensional flow in the $(\hat{x}, \hat{y})$ plane, $\vec{\omega}$ is parallel to $\hat{z}$. In fact, we can regard $\vec{\omega}$ as a scalar

$$\vec{\omega} = \omega \hat{z}$$

Now observe that

$$\vec{v}\times(\vec{\nabla}\times\vec{v}) = v^i\,\vec{\nabla}v^i - (\vec{v}\cdot\vec{\nabla})\,\vec{v}$$

$$= \tfrac{1}{2}\vec{\nabla}v^2 - (\vec{v}\cdot\vec{\nabla})\,\vec{v}$$

Then

$$\nabla\times\left[(\vec{v}\cdot\vec{\nabla})\vec{v}\,\right] = -\,\vec{\nabla}\times(\vec{v}\times\vec{\omega})$$

$$= -\nabla^i(\vec{v}\,\omega^i) + \vec{\nabla}^i(v^i\vec{\omega})$$

$$= -(\vec{\omega}\cdot\vec{\nabla})\vec{v} - \vec{v}(\vec{\nabla}\cdot\vec{\omega}) + (\vec{\nabla}\cdot\vec{v})\vec{\omega} + (\vec{v}\cdot\vec{\nabla})\vec{\omega}$$

The first three terms in the last line vanish, since $\vec{v}$ is independent of z, the div of a curl is zero, and the fluid is incompressible. Then what remains is

$$\vec{\nabla}\times((\vec{v}\cdot\vec{\nabla})\vec{v}) = (\vec{v}\cdot\vec{\nabla})\,\vec{\omega}$$

Then taking the curl of the Navier-Stokes equation gives

$$(\vec{v}\cdot\vec{\nabla})\vec{\omega} = \nu\,\nabla^2\vec{\omega}$$

of which only the $\hat{z}$ component is nonzero.

To enforce $\vec{\nabla}\cdot\vec{v} = 0$, I will introduce the *stream function.* In our discussion of potential flow, I called this function χ, but here it is typically called ψ. So, let $\psi(x,y)$ be such that

$$V_x = \frac{\partial \psi}{\partial y} \qquad V_y = -\frac{\partial \psi}{\partial x}$$

Then, automatically,

$$\frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y = \vec{\nabla} \cdot \vec{V} = 0$$

and

$$\omega = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi = -\nabla^2 \psi$$

Finally, extract the dimensionful parameters L, V_0,

$$x = L x' \qquad V = V_0 V' \qquad \psi = V_0 L \psi'$$

Inserting these definitions and then dropping the primes, we have

$$\left(V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y}\right) \nabla^2 \psi = \frac{1}{R} \nabla^4 \psi$$

$$(\partial_y \psi \, \partial_x - \partial_x \psi \, \partial_y) \nabla^2 \psi = \frac{1}{R} \nabla^4 \psi$$

where

$$R = \frac{V_0 L}{\nu}$$

The boundary conditions that must be satisfied are

$$\left.\begin{array}{l} v_y = 0 \\ v_x = 0 \end{array}\right\} \text{ at } y = 0$$

$$\vec{v} \to (1,0) \quad \text{as} \quad y \to \infty$$

Expressed on ψ, these boundary conditions become

$$\psi = 0 \quad \text{at } y = 0 \qquad (\text{implies } \nabla_x \psi = 0)$$

$$\nabla_y \psi = 0 \quad \text{at } y = 0$$

$$\psi \to y \qquad \text{as } y \to \infty$$

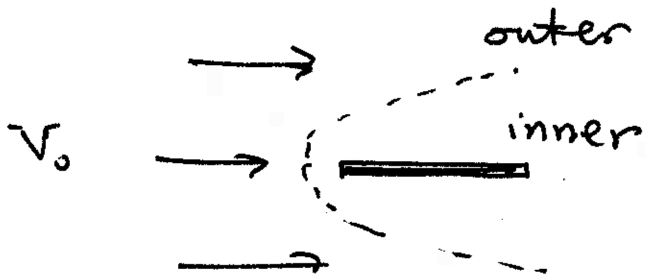
If we ignore the viscosity and the boundary condition at $y = 0$, we can easily see that the problem is solved by

$$\psi = y$$

More generally, ignoring the viscosity gives the equation

$$(\vec{V} \cdot \vec{\nabla})\, \omega = 0$$

that is, ω is *constant* along streamlines. If there is no vorticity in the flow in the region to the extreme left—and this is a property of our condition at $x \to -\infty$—then as long as this equation holds, no vorticity can be generated along the flow. The problem then divides into two distinct regions



In the outer region, we have ideal irrotational flow. The velocity in this region is then $(v_x, v_y) = (1, 0)$ after the rescaling above. However, there is a thin inner region in which the second derivatives with respect to y become large, viscosity is important, and this viscosity generates nonzero vorticity.

I will now solve for the fluid motion in the inner region. To do this, it is very convenient to further rescale y by a factor Δ, the thickness of this region

$$y = \Delta \cdot \mathbb{y} \qquad \frac{d}{dy} = \frac{1}{\Delta} \frac{d}{d\mathbb{y}}$$

For a smooth match to the outer region, v_x should be of order 1 in the inner region. Thus, we need $\nabla_{\mathbb{y}} \psi$ of order 1 or

$$\psi = \Delta \cdot \Psi(x, \mathbb{y})$$

$$v_y = -\frac{\partial}{\partial x} \psi = -\Delta \, \nabla_x \Psi = \mathcal{O}(\Delta)$$

Now rescale the Navier-Stokes equation and pick out the piece of each term that has the leading behavior for Δ small,

$$\nabla_y \psi \, \nabla_x \nabla^2 \psi = \frac{\Delta^2}{\Delta^3} \nabla_y \Psi \, \nabla_x \nabla_y^2 \Psi + \cdots$$

$$\nabla_x \psi \, \nabla_y \nabla^2 \psi = \frac{\Delta^2}{\Delta^3} \nabla_x \Psi \, \nabla_y^3 \Psi + \cdots$$

$$\frac{1}{R} \nabla^4 \psi = \frac{1}{R} \frac{\Delta}{\Delta^4} \nabla_y^4 \Psi$$

These are all of the same order if we take $R\Delta^2 = 1$. Thus, we recover the result for the thickness of the boundary layer

$$\Delta = \frac{1}{\sqrt{R}}$$

in agreement with Prandtl's argument given above.

These leading terms in the Navier-Stokes equation are

$$\nabla_y \Psi \, \nabla_x \nabla_y^2 \Psi - \nabla_x \Psi \, \nabla_y^3 \Psi = \nabla_y^4 \Psi$$

This equation can be rewritten

$$\nabla_y \left[\nabla_y^3 \Psi + \nabla_x \Psi \, \nabla_y^2 \Psi - \nabla_y \Psi \, \nabla_x \nabla_y \Psi \right] = 0$$

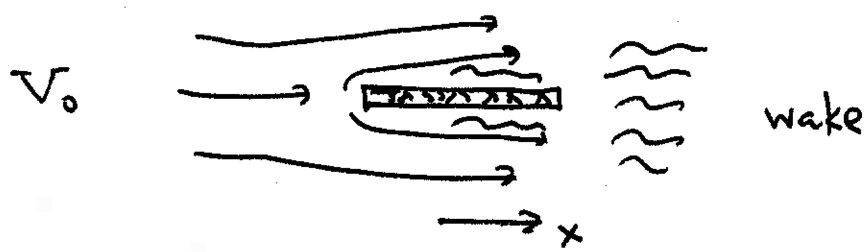
that is, the quantity

$$\nabla_y^3 \Psi + \nabla_x \Psi \, \nabla_y^2 \Psi - \nabla_y \Psi \, \nabla_x \nabla_y \Psi$$

is a function of x only. We can evaluate this function at large y where $\psi \to y$. There the expression is zero, and so it must vanish at all y. This gives a third-order equation

$$\nabla_y^3 \Psi + \nabla_x \Psi \, \nabla_y^2 \Psi - \nabla_y \Psi \, \nabla_x \nabla_y \Psi = 0$$

This equation is first-order in x, so we can imagine solving it by integrating from the left in x. Thus, the fluid motion in the wake $x > 1$ does not affect the form of the motion in the region next to the plate.



This third-order equation is invariant to the rescaling

$$\Psi \to \lambda \Psi \qquad y \to \lambda y \quad \text{or} \quad \nabla_y \to \frac{1}{\lambda} \nabla_y$$

$$x \to \lambda^2 x \quad \text{or} \quad \nabla_x \to \frac{1}{\lambda^2} \nabla_x$$

that is, each term in the equation is of order λ^{-2}. *Blasius* suggested that we look for a solution with this scaling property. This corresponds to the ansatz

$$\Psi = \sqrt{2x}\, f(w) \qquad w = \frac{y}{\sqrt{2x}}$$

Notate $df/dw = f'$, $d^2f/dw^2 = f''$, etc. Then we can work out the equation that the function $f(w)$ must satisfy.

$$\nabla_y^3 \Psi = \sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)^3 f''' = \frac{1}{2x} f'''$$

$$\nabla_x \Psi\, \nabla_y^2 \Psi = \sqrt{2x}\left[\frac{1}{2x} f - \frac{y}{\sqrt{2x}}\frac{1}{2x} f'\right] \cdot \sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)^2 f''$$

$$= \frac{1}{2x}\left[f f'' - w f' f''\right]$$

$$\nabla_y \Psi\, \nabla_x \nabla_y \Psi = \left(\sqrt{2x}\, \frac{1}{\sqrt{2x}} f'\right) \sqrt{2x} \left(\frac{1}{2x}\frac{1}{\sqrt{2x}} f' - \frac{1}{\sqrt{2x}}\frac{1}{2x} f' - \frac{1}{\sqrt{2x}}\frac{1}{2x}\frac{y}{\sqrt{2x}} f''\right)$$

$$= \frac{1}{2x}\left(- w f' f''\right)$$

In all, the equation

$$\nabla_y^3 \Psi + \nabla_x \Psi \nabla_y^2 \Psi - \nabla_y \Psi \nabla_x \nabla_y \Psi = 0$$

becomes

$$f''' + ff'' = 0$$

the *Blasius equation.*

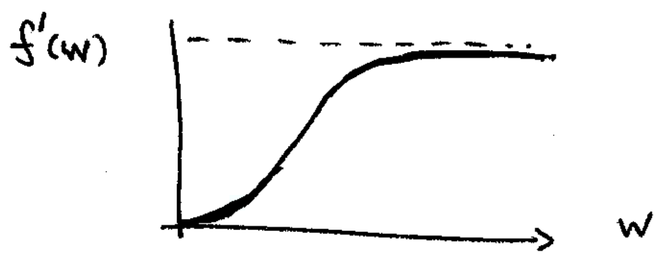
The boundary conditions on ψ are

$$\psi(x,0) = \frac{\partial}{\partial y}\psi(x,0) = 0 \qquad \psi \to y \text{ as } y \to \infty$$

On $f(w)$, these become

$$f(w=0) = f'(w=0) = 0 \qquad f' \to 1 \text{ as } w \to \infty$$

The solution to the Blasius equation is not a well-known special function, but one can easily find it numerically by *shooting*: choose a value of $f''(w=0)$, integrate forward, and then adjust the initial value to find one for which $f'(w) \to 1$ as $w \to \infty$.



The result of this process is the *Blasius function.* The properties of this function are near $w=0$ $\quad f(w) = \frac{1}{2}\alpha w^2 + \cdots \qquad \alpha = 0.4696$

near $w=\infty$ $\quad f(w) = w - \beta + \cdots \qquad \beta = 1.217$

We can find the asymptotic behavior of the Blasius function by plugging the latter expansion into the differential equation. Write

$$f(w) = w - \beta + \delta f(w)$$

Then

$$\delta f''' + (w-\beta)\,\delta f'' = 0$$

or

$$\delta f \sim \exp\left[-(w-\beta)^2/2\right]$$

We see that the corrections to the behavior $f(w) \approx w - \beta$ are exponentially small. We can then compute the velocity field as we exit the inner region where viscosity is important Putting all of the pieces together,

$$\psi(x,y) = \Delta \cdot \sqrt{2x}\, f\left(\frac{y}{\sqrt{2x}\,\Delta}\right)$$

$$\xrightarrow[y\to\infty]{} \Delta\sqrt{2x}\left(\frac{y}{\sqrt{2x}\,\Delta} - \beta\right)$$

$$= y - \sqrt{\frac{2x}{R}}\,\beta$$

Thus, the solution goes properly to the uniform flow at finite y. There is a small correction with nonzero vorticity, including a flow in the $\hat{y}$ direction. This is called the *displacement current.*

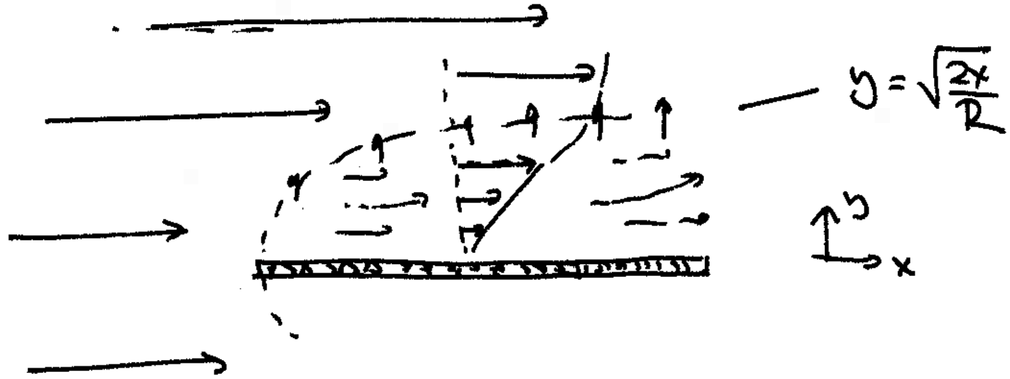
$$v_y = -\frac{\partial}{\partial x}\psi = +\frac{1}{\sqrt{2xR}}\,\beta$$

The boundary layer pushes some of the fluid in the outside laminar flow out of the way.

The transition betweeen the inner and outer regions takes place at value $w \sim 1$, that is

$$y \cong \sqrt{\frac{2x}{R}}$$

The boundary of this region is a parabola. Inside this parabola, v_x has the Blasius profile. The full picture of the flow is



Finally, we should compute the force on the plate. By dimensional analysis,

$$\frac{F_x}{\text{length in } z} = C_D\, \rho V_0^2 L$$

where C_D is a dimensionless coefficient called the *drag coefficient.* To compute F_x from our solution, we write

$$\frac{F_x}{\ell_z} = \underset{\substack{\uparrow \\ \text{(both sides)}}}{2} \int_0^L dx \;\; \eta \, \frac{\partial v_x}{\partial y}\bigg|_{y=0}$$

Going to dimensionless variables, this becomes

$$C_D = \frac{2}{R} \int_0^1 dx \;\; \frac{\partial v_x}{\partial y}\bigg|_0$$

After the second rescaling to the inner region

$$C_D = \frac{2}{R} \int_0^1 dx \;\; \frac{\Delta}{\Delta^2} \, \frac{d^2}{dY^2} \Psi \bigg|_{Y=0}$$

This becomes

$$C_D = \frac{2}{\sqrt{R}} \int_0^1 dx \, \frac{\sqrt{2x}}{(\sqrt{2x})^2} f''(w=0)$$

$$= \sqrt{\frac{2}{R}} \int_0^1 dx \, \frac{1}{\sqrt{x}} \, f''(0)$$

and so finally

$$C_D = \frac{2\sqrt{2}}{\sqrt{R}} \alpha = \frac{1.328}{\sqrt{R}}$$

It is quite remarkable that the drag force is proportional to $\sqrt{\eta}$ rather than η as $\eta \to 0$. This is the effect of the thinness of the boundary layer. Note that the force per unit area on the plate formally goes to infinity as we approach the front tip of the plate. However, this is an integrable singularity, so we do not need a more sophisticated treatment to obtain the force accurately.