Jan. 5

The Basic Equations of Fluid Dynamics

In fluid dynamics, we represent fluids using a continuum description. As physicists, we know that fluids are made of atoms. But, at distances much larger than the atomic size, we can represent the behavior of a fluid by collective macroscopic variables. Some examples of these are:

$$\rho = \text{mass}/\text{cm}^3$$

$$\vec{v} = \text{local velocity} \qquad \text{cm/sec}$$

$$\mathcal{E} = \text{energy density} \qquad \text{ergs}/\text{cm}^3$$

The evolution of these variables is governed by some equations of motion. A general principle for finding these equations is to examine the *conservation laws* that govern the system we are studying. We write the equations that represent the local conservation of basic quantities, then we try to add auxiliary relations in such a way as to obtain a closed set of partial differential equations (pde's).

Here is a simple illustration of this process. Consider the diffusion of a trace quantity in an immobile medium. Let $\mathbb{C}$ be the total number of atoms of the tracer in the sample. Represent the local number of atoms as a continuum density

$$\mathbb{C} = \int d^3x \; c(x,t) \qquad c(x,t) = \text{atoms}/\text{cm}^3$$

Then

$$\frac{d}{dt}\mathbb{C} = 0$$

is conserved. The density $c(\vec{x},t)$ is not constant, because atoms can move from one place to another. To express this, we need to define

$$\hat{n}\cdot\vec{j}_c(\vec{x},t) \qquad \text{flux of atoms}$$

a quantity equal to the number of atoms per cm^2 per second crossing an area normal to a unit vector $\hat{n}$. The components of $\hat{n}\cdot\vec{j}_c$ associate with three normal directions naturally form a vector.

For a fixed small volume, the change of the number of atoms within the volume is accounted for by the flow through the walls,

$$\frac{\partial}{\partial t}\int_V d^3x \; c(\vec{x},t) = -\int_{\partial V} d^2s \; \hat{n}\cdot\vec{j}_c(\vec{x},t)$$

where V is the volume and ∂V is the boundary of V. Using the divergence theorem

$$-\int d^2s \; \hat{n}\cdot\vec{j}_c = -\int d^3x \; \vec{\nabla}\cdot\vec{j}_c$$

If $c(\vec{x},t)$ is smooth, we can take V to be very small, compare the integrands, and find

$$\frac{\partial}{\partial t}c = -\vec{\nabla}\cdot\vec{j}_c$$

This is an exact relation. However, it is not useful by itself, because it expresses the dynamics of c in terms of a new set of quantities $\vec{j}_c$. To work with this equation, we need to related $\vec{j}_c$ back to a function of c. To do this, we need to make an approximation based on the smoothness of the continuum description. First, in a uniform sample, we do not expect atoms to flow to make their density non-uniform. So

$$\vec{j}_c = 0 \quad \text{if} \qquad c(x) = \text{const.}$$

Then if $\vec{j}_c$ is smooth as a function of $\vec{x}$, we can expand it in terms of derivatives of $c(\vec{x})$,

$$\vec{j}_c = -D\vec{\nabla}c + E\vec{\nabla}\nabla^2 c + Fc\vec{\nabla}c + \ldots$$

For a *dilute* species, terms nonlinear in c will be unimportant. For a density of atoms that changes slowly on the atomic scale, terms with higher derivatives will be unimportant,

$$\frac{E\nabla^2}{D} \sim \frac{\ell^2}{L^2}$$

where ℓ is the atomic size or the mean free path of tracer atoms in the medium and L is the length scale on whch c varies significantly. For typical situations

$$\sim \left(\frac{\text{Å}}{\text{mm}}\right)^2$$

Then it is enough to keep only the leading term in the expansion,

$$\vec{j}_c = -D\vec{\nabla}c$$

The parameter D is called the diffusion constant. It has the units of cm^2/sec. It is estimated by

$$D \sim (\text{mean free path}) \cdot (\text{thermal velocity})$$

Such a representation for the current $\vec{j}_c$ in terms of derivatives of the basic density, justified phenomenologically, is called a *constituitive equation.*

Putting this approximation for $\vec{j}_c$ back into the conservation law, we obtain a closed pde,

$$\frac{\partial c}{\partial t} = D \nabla^2 c$$

the *diffusion equation.* This equation is first-order in time, so, given an initial condition $c_0(\vec{x})$, we can integrate it forward to obtain $c(\vec{x}, t)$. Notice how the first-order equation we have found here contrasts with the second-order equations that we find in particle mechanics. The representation of $\vec{j}_c$ by a derivative of c is effectively an averaging procedure that removes information. The equation we obtain will be *dissipative*, that is, allowing the erasure of information.

The diffusion equation is a *linear pde*, so it is easy to solve. I presume that you have seen this solution before. If $c_0(\vec{x})$ is taken to be a delta function

$$c_0(\vec{x}) = \delta^{[3]}(x - y)$$

then the solution that evolves from it is a Gaussian

$$c(x,t) = \frac{1}{[4\pi D t]^{3/2}} e^{-|\vec{x}-\vec{y}|^2/4Dt}$$

You can easily see that this function, first, is normalized to 1 at all times, second, tends to a delta function as $t \to 0$, and third, satisfies the diffusion equation at any $t > 0$. The width of the Gaussian is

$$\sigma = [2Dt]^{1/2}$$

If we estimate $D =$ (velocity of particles) $\cdot$ (mean free path), then this width is

$$[2Dt]^{1/2} \sim \left[\frac{vt}{MFP}\right]^{1/2} \cdot MFP \sim N^{1/2} \cdot (MFP)$$

where N is the number of steps of a random walk taken by a particle in time t.

If we represent an arbitrary initial condition as

$$c_0(\vec{x}) = \int d^3y \; c_0(\vec{y}) \; \delta^{(3)}(\vec{x} - \vec{y})$$

then the solution of the diffusion equation for this initial condition is

$$c(x,t) = \int d^3y \; c_0(\vec{y}) \; \frac{1}{[4\pi Dt]^{3/2}} \, e^{-|x-y|^2/4Dt}$$

It is remarkable that, for this pde, I have written the most general solution in a closed form. Now—this is important—this is never going to happen again in this course. When we study flowing fluids, the pde's that we encounter will be nonlinear. They will have some simple solutions, but their general solutions will be very complex. You might think this is a bad thing, but actually it is a good thing, since it will lead to many fascinating phenomena that we will study in this course.

I would now like to generalize this diffusion equation written above to the equation for diffusion of a tracer in a moving medium. Again we can analyze a small volume and write the equation that the change in the number of atoms in the volume is given by the flow of atoms through the walls. It is little more confusing, though, to write this equation taking into account the motion of the medium. We have two distinct choices. We can consider a volume fixed in space, or we can consider a volume fixed in the fluid as it moves. These viewpoints have names:

- The *Eulerian viewpoint* is to follow the dynamics at a definite point in space.
- The *Lagrangian viewpoint* is to follow the dynamics by considering an definite fluid element that might be moving from point to point.

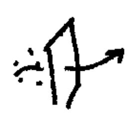
It is often easier to derive the correct equations using the Lagrangian point of view, but it is usually easier to solve the equations using the Eulerian point of view. I would like to derive the diffusion equation in a moving fluid from both viewpoints for comparison.

Begin with the Eulerian approach to this derivation. The constituitive equation

$$\vec{j}_c = -D\vec{\nabla}c$$

will give the flow of atoms in the frame in which the fluid is locally at rest. If we consider a volume V fixed in space, the flow of atoms across the walls is given by a combination of two effects, the current $\vec{j}_c$ that responds to density gradients in the fluid and the flow of atoms through the walls as they follow the fluid flow. Then the change in the number of atoms in the fixed volume is

$$\frac{d}{dt}\int_V d^3x\, c = -\int d^2s\, \hat{n}\cdot\vec{j}_c - \int d^2s\, \hat{n}\cdot\vec{v}c$$



We can write this as an integral over the volume,

$$\int_V d^3x \, \frac{\partial c}{\partial t} = - \int_V d^3x \left[\vec{\nabla} \cdot \vec{j}_c + \vec{\nabla} \cdot (\vec{v} c) \right]$$

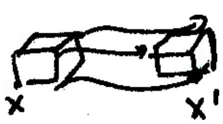
Then, taking the volume to be small, we have locally

$$\frac{\partial c}{\partial t} + \vec{\nabla} \cdot (\vec{v} c) = D \nabla^2 c$$

Here is the Lagrangian viewpoint. If the fluid is moving with a local velocity $\vec{v}(\vec{x})$, small volume $V(0)$ will deform to $V(\Delta t)$. The number of atoms inside the volume will be changed only by a flow $\vec{j}_c$ across the walls. Then the change in the number of atoms in the volume over a time Δt is given by

$$\Delta(\text{atoms}) = \int_{V'} d^3x' \, c(x', t+\Delta t) - \int_V d^3x \, c(x,t) = - \int d^2s \, \hat{n} \cdot \vec{j}_c \, \Delta t$$

where



$$\vec{x}' = \vec{x} + \vec{v}(x,t) \Delta t$$

It is useful to change variables from $\vec{x}'$ to $\vec{x}$. This requires the Jacobian

$$\left| \frac{\partial x'^i}{\partial x^j} \right| = \det \left(\delta^{ij} + \frac{\partial}{\partial x^j} v^i \Delta t \right) \approx 1 + \frac{\partial}{\partial x^i} v^i \Delta t$$

Now, assuming smooth behavior, we can write all of the quantities in the equation as integrals d^3x over the original volume $V(0)$

$$\int d^3x \quad (1 + \vec{\nabla}\cdot\vec{v}\,\Delta t)\left(c(x,t) + \Delta t\,\frac{\partial c}{\partial t} + \Delta t\,\vec{v}\cdot\vec{\nabla} c\right)$$

$$= \quad - \int d^3x \;\; \vec{\nabla}\cdot(-D\vec{\nabla}c) \quad \Delta t$$

Then we find the diffusion equation in the form

$$\frac{\partial c}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,c \quad = \quad -(\vec{\nabla}\cdot\vec{v})\,c \quad + D\,\nabla^2 c$$

$$\frac{D}{Dt}\,c \quad = \quad -(\vec{\nabla}\cdot\vec{v})c \quad + D\,\nabla^2 c$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$$

This object is called the *convective derivative.* Moving the term with $\vec{\nabla}\cdot\vec{v}$ to the left-hand side, we find the equation derived in the previous paragraph.

If $\vec{v}(\vec{x},t)$ is known, this diffusion equation is a linear pde for $c(\vec{x},t)$, and sometimes it can be solved exactly. However, if we also need to solve for $\vec{v}(\vec{x},t)$ in a way that is coupled to $c(\vec{x},t)$, we have a nonlinear problem.

We will now discuss the conservation laws for the most basic conserved quantities of a simple fluid—mass, momentum, and energy. Begin with the equation for the conservation of mass. In these lectures, I will distinguish between the mass density $\rho(\vec{x},t)$, in g/cm^3, and the particle number density $n(\vec{x},t)$, in atoms/cm^3. The conservation law for mass will involve ρ and also a mass current,

$$\vec{J} = \rho\vec{v}$$

giving the rate of transport of mass across a surface of unit area. If we define an Eulerian volume and integrate the mass density over this volume, that quantity can change only if mass is transported through the boundary by the flow. Then

$$\frac{d}{dt}\int_V d^3x \; \rho = -\int_{\partial V} d^2s \; \hat{n}\cdot\vec{j} = -\int_V d^3x \; \vec{\nabla}\cdot\vec{j}$$

For a small volume V, we find

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j} = -\vec{\nabla}(\rho\vec{v})$$

This is the *equation of continuity.* It can also be written as

$$\frac{D\rho}{Dt} = -(\vec{\nabla}\cdot\vec{v})\,\rho$$

From the Lagrangian viewpoint, the mass inside a box moving with the fluid does not change, but the mass density can change to compensate changes in the volume of the box. From the Jacobian computed above, we see that the rate of change of this volume is just

$$\frac{d\,(\text{Vol})}{dt} = -(\vec{\nabla}\cdot\vec{v})\cdot(\text{Vol})$$

In the first weeks of this course, I will often simplify to the case of an *incompressible* fluid. For such an idealized fluid,

$$\rho = \text{constant} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{v} = 0$$

Notice that $\vec{\nabla} \cdot \vec{v} = 0$ is a constraint imposed at fixed time. If the fluid is deformed, the effect of this constraint propagates instantaneously. Any real fluid is compressible to some extent, and the effects of deformations propagate at the speed of sound. The approximation that a fluid is incompressible is then generally valid only for motions that take place at velocities much slower than the speed of sound.

Next, consider the conservation of momentum. The density of momentum in a fluid is mass density times velocity

$$\mathcal{P}^i = \rho v^i$$

and equal to the mass current. In the Lagrangian viewpoint, the change in momentum of a fluid element is given by the forces that act on that fluid element. The change in momentum in a time Δt is

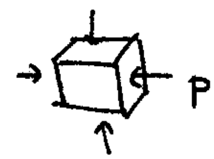
$$\int d^3x' \, \mathcal{P}^i(x', t+\Delta t) - \int d^3x \, \mathcal{P}^i(x,t) = \int d^3x \, (\text{forces}) \, \Delta t$$

The forces that act on the fluid element are of two types. First, there are external forces, for example, the force of gravity or electrostatic forces. In the derivation here, I will keep the force of gravity explicitly. Let the gravitational potential be $\Phi(\vec{x})$. Then the gravitational force per unit volume is

$$-\rho \vec{\nabla} \Phi$$

Beyond this, each small element of fluid is pushed on by its neighboring elements. Each element of fluid exerts a definite force per unit area on the fluid just outside its boundary. This is the local pressure $p(\vec{x}, t)$. In a fluid without internal structure, the pressure must be isotropic, the same in all directions. Then the total pressure force that is applied to a fluid element by its neighbors is

$$-\int_{\partial V} d^2s \ \hat{n} \ p(x,t) \quad = \quad -\int_V d^3x \ \vec{\nabla} p$$



There are other internal forces in a fluid. For example, layers of fluid that rub against one another will apply friction forces. I would like to neglect this friction for the moment; we will study it systematically a few lectures from now.

Assembling these pieces, we have the equation

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{v}\cdot\vec{\nabla}(\rho\vec{v}) + (\vec{\nabla}\cdot\vec{v})\rho\vec{v} = -\vec{\nabla}p - \rho\vec{\nabla}\Phi + (\text{friction})$$

Simplifying by the use of the equation of continuity, this becomes

$$\rho\frac{\partial}{\partial t}\vec{v} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p - \rho\vec{\nabla}\Phi + \cdots$$

$$\rho\frac{D}{Dt}\vec{v} = -\vec{\nabla}p - \rho\vec{\nabla}\Phi$$

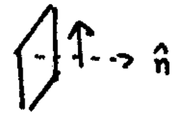
which is called *Euler's equation.* This is the basic equation of motion for the frictionless fluid, the replacement for Newton's law for a continuum system.

We can also derive this equation from the Eulerian point of view. To begin, write the equation for the change in momentum of a fluid in a fixed Eulerian volume,

$$\frac{d}{dt}\int_V d^3x\, \mathcal{P}^i = -(\text{momentum outflow})$$

The right-hand side of this equation includes external forces, internal forces, and changes in momentum due to fluid moving through the wall of the volume, which is also a sort of force acting on the fluid in the volume. I would like to group these last two effects together and define an object T^{ij} such that

$$T^{ij}\hat{n}^j = \text{momentum}/(\text{m}^2\,\text{sec}) \text{ flowing across a surface} \perp \text{to } \hat{n}$$



This is called the *stress tensor.* It has two indices. A component T^{ij} of the stress tensor represents the momentum in the i direction flowing, per unit area per unit time, across a surface oriented perpendicular to the j direction

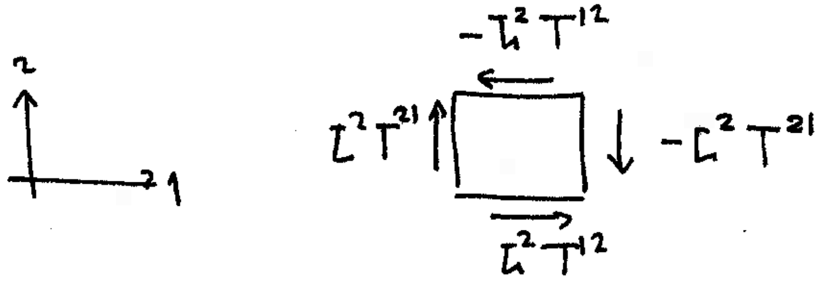
$$-(\text{momentum outflow}) = -\int_{\partial V} d^2s\, \hat{n}^k T^{ik} = \int_V d^3x\, \nabla^k T^{ik}$$

Then the equation of momentum conservation in a fluid is

$$\frac{\partial}{\partial t}(\rho v^i) = -\nabla^k T^{ik} + (\text{external forces})$$

where $\mathcal{P}^i = \rho v^i$.

I will now argue that T^{ij} must be a *symmetric* tensor. Consider, for example, the consequences of $T^{12} \neq T^{21}$, for a small cube in the fluid of size L. The neighboring elements of the fluid exert a force on the surface normal to $\hat{2}$ of magnitude $-L^2T^{12}$ in the $\hat{1}$ direction and an opposite force on the opposite face of the cube. They also exert a force on the surface normal to $\hat{1}$ of magnitude $-L^2T^{21}$ in the $\hat{2}$ direction and an opposite force on the face opposite to this one. The full set of forces is



If $T^{12} \neq T^{21}$, there is a net torque on the cube. Worse, for a small cube, the torque is of order L^3, but the moment of inertia is only of order L^5 as $L \to 0$. So every small element of fluid would spin up to high angular velocity, which is untenable, unless $T^{12} = T^{21}$.

We must determine T^{ij} by phenomenological analysis. First, consider an isotropic fluid at rest. Here, T^{ij} can only have the form

$$T^{ij} = A\,\delta^{ij}$$

since δ^{ij} is the only symmetric invariant. The force exerted per unit area is just A, and so we can identify this quantity with the pressure p.

If there is a fluid flow with a local velocity $\vec{v}$, we can determine a part of the stress tensor by making a Galilean boost, or, alternatively, by considering the transport of momentum density at the flow velocity. Either way, this gives

$$T^{ij} = p\,\delta^{ij} + \rho v^i v^j$$

In a later lecture, I will add further terms to T^{ij} to represent the effect of internal friction in the fluid.

With the form of T^{ij} just given, the equation of conservation of momentum in the fluid takes the form

$$\frac{\partial}{\partial t}(\rho v^i) = -\nabla^k(p\delta^{ik} + \rho v^i v^k) - \rho\nabla^i\Phi$$

We can rewrite this as

$$v^i\frac{\partial\rho}{\partial t} + \rho\frac{\partial v^i}{\partial t} = -\nabla^i p - v^i\vec{\nabla}^k(\rho v^k) - (\vec{v}\cdot\vec{\nabla})v^i - \rho\nabla^i\Phi$$

or, cancelling terms that are equal by the equation of continuity,

$$\rho\frac{\partial}{\partial t}v^i + (\vec{v}\cdot\vec{\nabla})v^i = -\nabla^i p - \rho\nabla^i\Phi$$

This gives another derivation of Euler's equation.

Note that Euler's equation plus the equation of continuity does not by itself give a well-defined problem to solve. It is a set of four equations for five unknowns, the three components of v^i, the density ρ, and the pressure p. To obtain a well-posed system of equations, we need an additional equation that gives the pressure in terms of other variables. For this purpose, we could use the *equation of state* that gives the pressure in terms of the density (at an assumed constant temperature). Alternatively, we could add the constraint that the fluid is incompressible and so the density is constant. Most of our examples in the next few weeks will use this latter strategy.

Before discussing the energy equation, I would like to make a small detour and discuss *hydrostatics*. In a static fluid, $\vec{v} = 0$ and so the equation of momentum conservation is very simple,

$$0 = -\vec{\nabla} p - \rho \vec{\nabla} \Phi$$

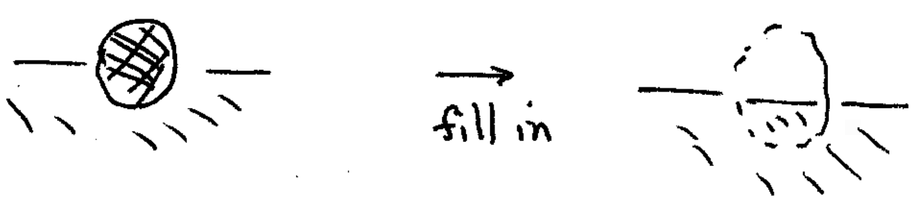
Then the pressure can be obtained at any point by integration. In particular, the pressure is constant along a surface on which the gravitational potential is constant, so for $\Phi = gz$, the pressure is also a function of z only. For a fluid of constant density, we have simply

$$p = p_0 - \rho g z$$

Even if the density varies, we have an explicit expression,

$$p = p_0 - g \int_0^z dz\, \rho(z)$$

A very beautiful result is hydrostatics is *Archimedes' principle*: The weight of a body that floats in water is equal to the weight of the water displaced. More generally, an object that is neutrally bouyant in a static fluid has the weight of the fluid it displaces. The first statement is obtained by applying the second statement to a fluid system with water and air in coexistence:



Here is the proof. For a neutrally bouyant object, the balance of gravitational and pressure forces is expressed by

$$0 = -Mg - \int_{\partial V} d^2s\, \hat{n} \cdot p$$

where $p(\vec{x})$ is the fluid pressure on the surface of the object. Now remove the object and fill the gap with fluid in a way that maintains the static situation. Notice that $p(\vec{x})$ on the surface does not change, since the value of $p(\vec{x})$ is a definite function of z determined far away from the object. If M_f is the mass of the fluid required to fill the gap,

$$0 = -M_f g - \int_{\partial V} d^2s\, \hat{n}\, p$$

Then,

$$M = M_f$$

Finally, I will discuss the conservation of energy in a fluid. The basic equation will be a conservation law similar to those we have encountered above,

$$\frac{\partial \rho_\epsilon}{\partial t} = -\vec{\nabla} \cdot \vec{j}_\epsilon$$

where ρ_ϵ is the energy density and $\vec{j}_\epsilon$ is the current that gives the flow of energy across a surface per cm^2 per second. We now need to find expressions for these quantities. In this discussion, I will account only the mechanical energy of the fluid. Then energy will be conserved only if external forces are time-dependent, if there is no friction, and if there are no other mechanisms for converting mechanical energy to heat. In this derivation, I will consider Φ to be independent of time and ignore the friction terms in T^{ij} For the last criterion, though, we need to recall some elements of thermodynamics.

In thermodynamics, the internal energy E of a system changes when work is done on it or when heat is added to it. This is expressed by the differential relation

$$dE = TdS - pdV$$

where S is the *entropy.* When pressure is held fixed, it is often more convenient to work with the enthalpy

$$H = E + pV$$

which satisfies

$$dH = TdS + Vdp$$

In a fluid, we can consider the density of E or H. It is conventional to define quantities $u(\vec{x},t)$, $h(\vec{x},t)$, and $s(\vec{x},t)$ as the energy, enthalpy, and entropy, respectively, per gram of fluid. Then

$$E = \int d^3x\, \rho u \qquad H = \int d^3x\, \rho h \qquad \text{etc.}$$

The quantity h obeys

$$h = u + \frac{p}{\rho} \qquad dh = T\,ds + \frac{1}{\rho}\,dp$$

The assumption that no energy is taken up in heating the fluid can then be expressed by the statement that there is no change in the entropy of the fluid along the flow

$$\frac{\partial}{\partial t}(\rho s) + \vec{\nabla}\cdot(\rho\vec{v}s) = 0$$

$$\text{or} \qquad \frac{D}{Dt}s = \frac{\partial}{\partial t}s + \vec{v}\cdot\vec{\nabla}s = 0$$

A fluid with fixed entropy and zero internal friction is called an *ideal fluid.* In this case

$$T^{ij} = p\,\delta^{ij} + \rho v^i v^j$$

exactly. The approximation of fixed entropy is typically a good one for flows that transfer fluid from place to place faster than heat can diffuse in or out.

For an ideal fluid, I claim that the equation of local energy conservation is satisfied with

$$\rho_\epsilon = \rho\left(\tfrac{1}{2}v^2 + u + \Phi\right)$$

$$\vec{j}_\epsilon = \rho\vec{v}\left(\tfrac{1}{2}v^2 + h + \Phi\right)$$

We can check this explicitly.

$$\frac{\partial}{\partial t} \rho \left(\tfrac{1}{2}v^2 + u + \Phi\right)$$

$$= \left(\frac{\partial \rho}{\partial t}\right)\left(\tfrac{1}{2}v^2 + u + \Phi\right) + \rho v^i \frac{\partial v^i}{\partial t} + \rho \frac{\partial u}{\partial t}$$

$$\vec{\nabla} \cdot \left(\rho \vec{v} \left(\tfrac{1}{2}v^2 + h + \Phi\right)\right)$$

$$= \vec{\nabla}\cdot(\rho\vec{v}) \left(\tfrac{1}{2}v^2 + h + \Phi\right) + \rho\vec{v}\; v^i \vec{\nabla} v^i + \rho \vec{v}\cdot\vec{\nabla} h + \rho \vec{v}\cdot\vec{\nabla}\Phi$$

The terms

$$\left[\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v})\right]\left(\tfrac{1}{2}v^2 + h + \Phi\right)$$

cancel out by the equation of continuity. Using also

$$\frac{\partial u}{\partial t} = T \frac{\partial s}{\partial t} - \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}$$

what remains from $\partial \rho_\epsilon / \partial t$ is

$$\frac{\partial \rho}{\partial t}\left(-\frac{P}{\rho}\right) + \rho v^i \frac{\partial v^i}{\partial t} + \rho \frac{\partial u}{\partial t}$$

$$= \rho T \frac{\partial s}{\partial t} + \rho v^i \left[-\vec{v}\cdot\vec{\nabla} v^i - \frac{1}{\rho}\nabla^i P - \nabla^i \Phi\right]$$

Using

$$\vec{\nabla} h = T\vec{\nabla} s + \frac{1}{\rho}\vec{\nabla} p$$

what remains from $\vec{\nabla}\cdot\vec{j}_\epsilon$ is

$$\rho v^i (\vec{v}\cdot\vec{\nabla}) v^i + \rho\vec{v}\, T\cdot\vec{\nabla} s + \rho\frac{\vec{v}\cdot}{\rho}\vec{\nabla} p + \rho\vec{v}\cdot\vec{\nabla}\Phi$$

$$= \rho T\, \vec{v}\cdot\vec{\nabla} s + \rho v^i \left[\vec{v}\cdot\vec{\nabla} v^i + \frac{1}{\rho}\vec{\nabla} p + \vec{\nabla}\Phi \right]$$

Now everything cancels if we use the ideal fluid condition

$$\frac{\partial}{\partial t} s + \vec{v}\cdot\vec{\nabla} s = 0$$

When we are considering ideal fluids, there is an interesting way to rewrite the Euler equation. Define the *vorticity*

$$\vec{\omega} = \vec{\nabla}\times\vec{v}$$

which measures the rotation in a fluid flow. From

$$\vec{A}\times(\vec{\nabla}\times\vec{B}) = A^i\,\vec{\nabla} B^i - (\vec{A}\cdot\vec{\nabla})\vec{B}$$

we can see that

$$\vec{v}\times\vec{\omega} = \vec{v}\times(\vec{\nabla}\times\vec{v})$$
$$= v^i\vec{\nabla}v^i - (\vec{v}\cdot\vec{\nabla})\vec{v}$$
$$= \tfrac{1}{2}\vec{\nabla}(v^2) - (\vec{v}\cdot\vec{\nabla})\vec{v}$$

This allows us to rewrite the $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term in the Euler equation to obtain

$$\frac{\partial\vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla}v^2 - \vec{v}\times\vec{\omega} + \frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\Phi = 0$$

up to friction terms that we ignore for an ideal fluid.

In a *steady flow*, a flow that does not depend on t,

$$\frac{\partial\vec{v}}{\partial t} = 0 \qquad \frac{\partial s}{\partial t} = 0 \Rightarrow \vec{v}\cdot\vec{\nabla}s = 0$$

The second equation here implies that

$$\vec{v}\cdot\vec{\nabla}h = \frac{1}{\rho}\vec{v}\cdot\vec{\nabla}p$$

for an ideal fluid. Then $\vec{v}$ dotted into the above equation gives a simple gradient

$$\vec{v}\cdot\vec{\nabla}\left(\tfrac{1}{2}v^2 + h + \Phi\right) = 0$$

This is *Bernoulli's Theorem.* The quantity

$$\mathcal{B} = \frac{1}{2}v^2 + h + \Phi$$

is conserved along lines of flow. In conservative particle mechanics, there is a trade-off between kinetic and potential energy. In an ideal fluid, Bernoulli's Theorem gives a similar trade-off between kinetic energy, gravitational energy, and pressure.