Physics 210 - Final Exam
Solutions

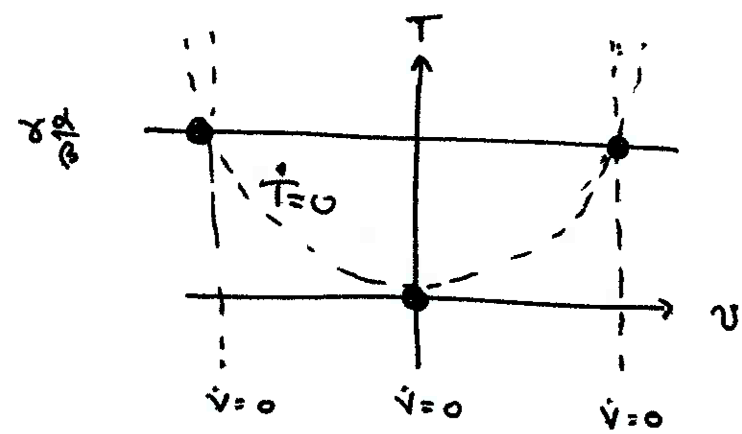
1.) a) The fixed points are the solutions of

$$0 = \alpha v - \beta v^3 \Rightarrow v = 0 \quad v = \pm\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}$$

$$0 = -T + \gamma v^2$$

so the fixed points are located at

$$(v,T) = (0,0)\ ,\ \left(\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}, \gamma\frac{\alpha}{\beta}\right),\ \left(-\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}, \gamma\frac{\alpha}{\beta}\right)$$



b.) Near $(v,T) = (0,0)$

$$\dot{v} = \alpha v$$
$$\dot{T} = -T$$

the eigenvalues and eigenvectors are

$$\lambda = +\alpha \qquad \xi = (1,0) \qquad \text{unstable}$$

$$\lambda = -1 \qquad \xi = (0\ 1) \qquad \text{stable}$$

near $(v,T) = (v_* \ T_*)$ $\qquad v_* = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \quad T_* = \gamma\frac{\alpha}{\beta}$

$$\Delta v = v - v_* \qquad \Delta T = T - T_*$$

$$\Delta\dot{v} = (\alpha - 3\beta v_*^2)\,\Delta v = -2\alpha\ \Delta v$$

$$\Delta\dot{T} = -\Delta T + 2\gamma v_* \Delta v$$

so

$$\begin{pmatrix}\Delta v \\ \Delta T\end{pmatrix}^{\cdot} = \begin{pmatrix}-2\alpha & 0 \\ 2\gamma v_* & -1\end{pmatrix}\begin{pmatrix}\Delta v \\ \Delta T\end{pmatrix}$$

$\xi = (0,1)$ is an eigenvector w. eigenvalue $\lambda = -1$

$\xi = (1, a)$ is an eigenvector w. eigenvalue $\lambda = -2\alpha$

if $\qquad 2\gamma v_* - a = -2\alpha\, a$

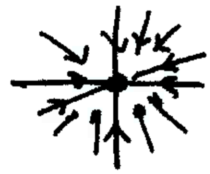
$$2\gamma v_* = (1-2\alpha)a$$

$$a = \left(\frac{2\gamma}{1-2\alpha}\right) v_*$$

so

$$\lambda = -1 \qquad \xi = (0,1)$$

$$\lambda = -2\alpha \qquad \xi = \left(1,\ \frac{2\gamma}{1-2\alpha}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\right)$$



(drawn for $1 > 2\alpha$)

near $\quad (v,T) = (-v_*, T_*) \qquad \Delta v = v + v_* \qquad v_* = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}$

$$\Delta\dot{v} = -2\alpha\,\Delta v$$

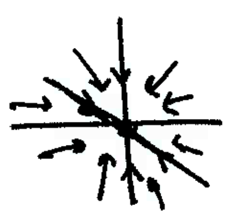
$$\Delta\dot{T} = -\Delta T - 2\gamma v_*\,\Delta v$$

so

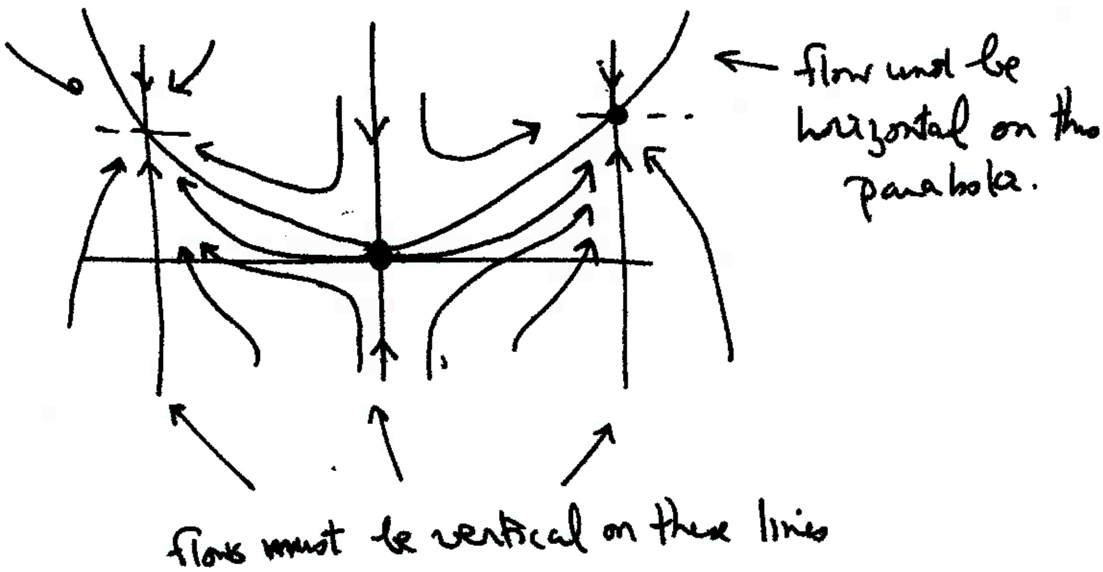
$$\lambda = -1 \qquad \xi = (0,1)$$

$$\lambda = -2\alpha \qquad \xi = \left(1,\ -\frac{2\gamma}{1-2\alpha}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\right)$$

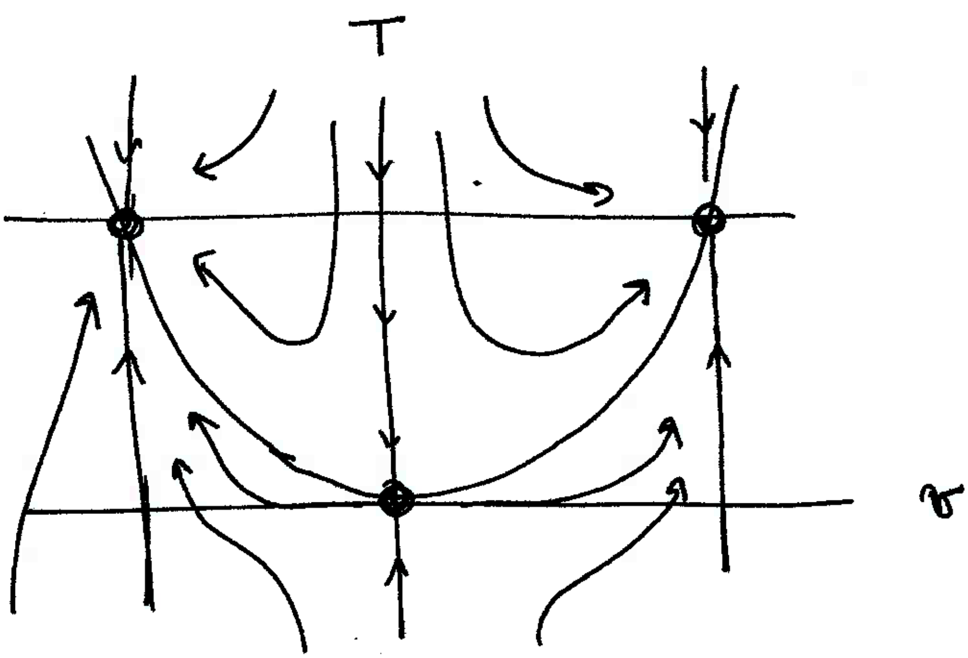
the mirror image of the previous case



going up these flows!

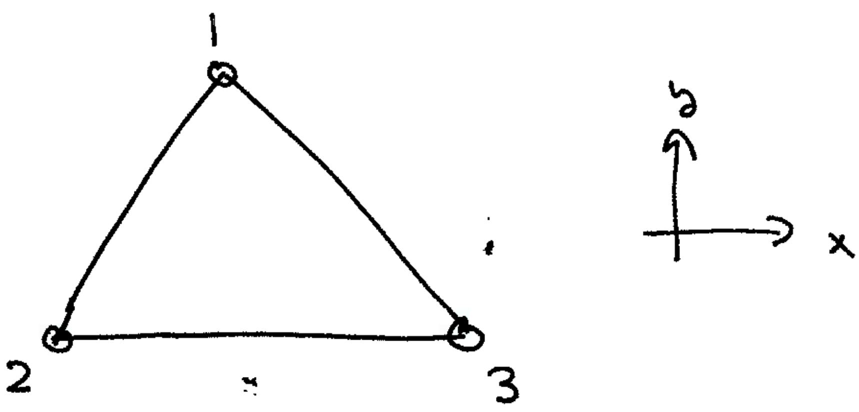


again

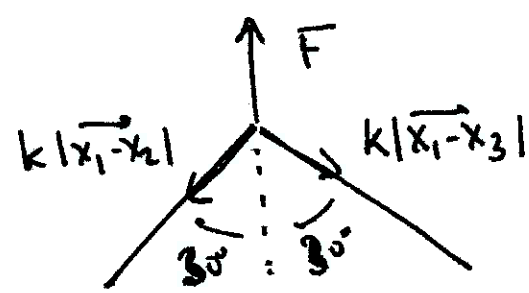


2.) a.)

For the configuration:



Forces on the vertex 1 (by symmetry, it suffices to consider 1 vertex only)



If the triangle is symmetric, the forces proportional to k balance in the $\hat{x}$ direction. In the $\hat{y}$ direction, the total force is

$$F - 2ka\cos 30^\circ = F - 2ka\frac{\sqrt{3}}{2}$$

so if $F = \sqrt{3}ka$ the three forces balance and the figure is in equilibrium.

b.) If a changes from the equilibrium value $a_* = \frac{F}{\sqrt{3}k}$, the force on the vertex 1 is

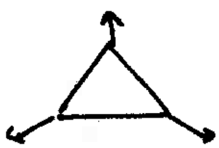
$$\vec{F} = F\hat{y} - \sqrt{3}ka\,\hat{y}$$

$$\vec{F} = -\sqrt{3}\, k\hat{y}\,(a - a_*)$$

This is a restoring force that decreases a if $a > a_*$ and increases a if $a < a_*$

c.) The deformation described in part (b) is

$$(\Delta x_1\ \Delta y_1\ \ \Delta x_2\ \Delta y_2\ \ \Delta x_3\ \Delta y_3)$$

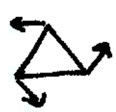


$$= \left(0\ \ 1,\ -\tfrac{\sqrt{3}}{2},\ -\tfrac{1}{2},\ +\tfrac{\sqrt{3}}{2},\ -\tfrac{1}{2}\right)\cdot\Delta$$

The balance of forces in the figure is not changed by

translation in x  $(1,0,1,0,1,0)\cdot\Delta$

translation in y  $(0,1,0,1,0,1)\cdot\Delta$

rotation about the center  $\left(-1,\ 0,\ \tfrac{1}{2}\ -\tfrac{\sqrt{3}}{2},\ \tfrac{1}{2},\ \tfrac{\sqrt{3}}{2}\right)\cdot\Delta$

d.) To solve the rest of the problem, we really have to look at Newton's laws. For the vertex 1

$$m\ddot{\vec{x}}_1 = -k\,(\overrightarrow{x_1 - x_2}) - k(\overrightarrow{x_1 - x_3}) + F\,\widehat{\left(x_1 - \tfrac{x_2+x_3}{2}\right)}$$

unit vector in this direction

$$m\ddot{\vec{x}}_1 = -k\,\vec{Z}_1 + F\,\hat{z}_1$$

where $\vec{Z}_1 = 2\vec{x}_1 - (\vec{x}_2 + \vec{x}_3)$

expand $\vec{Z}_1$ about the equilibrium:

$$\vec{Z}_1 = 2\cdot\frac{\sqrt{3}}{2}a\hat{y} + (2\Delta x_1 - \Delta x_2 - \Delta x_3,\ 2\Delta y_1 - \Delta y_2 - \Delta y_3)$$

$$|\vec{Z}_1| = (\vec{Z}_1^{\,2})^{1/2}$$

$$= \left[(\sqrt{3}a + 2\Delta y_1 - \Delta y_2 - \Delta y_3)^2 + O(\Delta x)^2\right]^{1/2}$$

$$= \sqrt{3}a + 2\Delta y_1 - \Delta y_2 - \Delta y_3$$

$$\hat{z}_1 = \frac{\vec{Z}_1}{|Z_1|} = \hat{y} + \frac{2\Delta x_1 - \Delta x_2 - \Delta x_3}{\sqrt{3}a}\,\hat{x}$$

so, to linear order in small deviations

$$m\Delta\ddot{x}_1 = -k\,(2\Delta x_1 - \Delta x_2 - \Delta x_3) + \frac{F}{\sqrt{3}a}(2\Delta x_1 - \Delta x_2 - \Delta x_3)$$

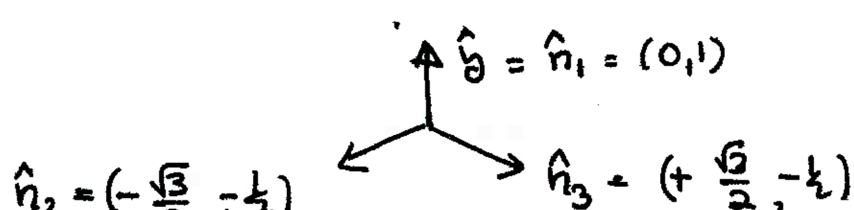
$$= 0$$

$$m\Delta\ddot{y}_1 = -k\,(2\Delta y_1 - \Delta y_2 - \Delta y_3)$$

again

$$m\Delta\ddot{\vec{x}}_1 = -k\hat{y}\,(2\Delta y_1 - \Delta y_2 - \Delta y_3)$$

for 2,3 rotate this equation appropriately



$$m\,\ddot{\vec{\Delta x}}_1 = -k\,\hat{n}_1\;\hat{n}_1\cdot(2\vec{\Delta x}_1-\vec{\Delta x}_2-\vec{\Delta x}_3)$$
$$m\,\ddot{\vec{\Delta x}}_2 = -k\,\hat{n}_2\;\hat{n}_2\cdot(2\vec{\Delta x}_2-\vec{\Delta x}_1-\vec{\Delta x}_3)$$
$$m\,\ddot{\vec{\Delta x}}_3 = -k\,\hat{n}_3\;\hat{n}_3\,(2\vec{\Delta x}_3-\vec{\Delta x}_1-\vec{\Delta x}_2)$$

Check the eigenvectors that we know already:

$$\vec{\Delta x}_i = \Delta\cdot\hat{n}_i$$

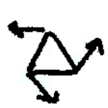
use $\hat{n}_i\cdot\hat{n}_i = 1$ $\quad\hat{n}_i\cdot\hat{n}_j = -\frac{1}{2}\quad i\neq j$

$$m\,\ddot{\Delta}\,\hat{n}_i = -k\hat{n}_i\left(2+\tfrac{1}{2}+\tfrac{1}{2}\right)\cdot\Delta$$

all 3 equations become $m\ddot{\Delta} = -3k\,\Delta$ stable w. $\omega^2 = \frac{3k}{m}$

$$2\vec{\Delta x}_1-\vec{\Delta x}_2-\vec{\Delta x}_3 = 0 \quad \text{etc.}$$

so all 3 equations give $m\ddot{\Delta} = 0$



in all cases $\vec{\Delta x}_i\cdot\hat{n}_i = 0$

$$(\vec{\Delta x}_j+\vec{\Delta x}_k)\cdot\hat{n}_i = 0 \quad j,k\neq i$$

so all 3 equations give $m\ddot{\Delta} = 0$

There are two more orthogonal directions in the space of eigenvectors

$$\eta_a = \left(0,1,+\tfrac{\sqrt{3}}{2},-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}\right)$$

$$\eta_b = \left(1,0,-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2},+\tfrac{\sqrt{3}}{2}\right)$$

then

$$m \ddot{\vec{\Delta x}}_1 = -k\, \hat{n}_1\, \hat{n}_1 \cdot (2\, \vec{\Delta x}_1 - \vec{\Delta x}_2 - \vec{\Delta x}_3)$$

$$m \ddot{\vec{\Delta x}}_2 = -k\, \hat{n}_2\, \hat{n}_2\, (2\, \vec{\Delta x}_2 - \vec{\Delta x}_1 - \vec{\Delta x}_3)$$

$$m \ddot{\vec{\Delta x}}_3 = -k\, \hat{n}_3\, \hat{n}_3\, (2\, \vec{\Delta x}_3 - \vec{\Delta x}_1 - \vec{\Delta x}_2)$$

This would look cleaner if we adopted coordinates parallel and perpendicular to the $\hat{n}_i$:

$$\vec{\Delta x}_1 = \Delta\alpha_1\, \overrightarrow{(0,1)} + \Delta\beta_1\, \overrightarrow{(1,0)}$$

$$\vec{\Delta x}_2 = \Delta\alpha_2 \left(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) + \Delta\beta_2 \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$

$$\vec{\Delta x}_3 = \Delta\alpha_3 \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) + \Delta\beta_3 \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right)$$

then

$$m\, \Delta\ddot{\alpha}_1 = -k \left[2\,\Delta\alpha_1 + \tfrac{1}{2}(\Delta\alpha_2 + \Delta\alpha_3) - \tfrac{\sqrt{3}}{2}(\Delta\beta_2 - \Delta\beta_3) \right]$$

$$m\, \Delta\ddot{\beta}_1 = 0$$

$$m\, \Delta\ddot{\alpha}_2 = -k \left[2\,\Delta\alpha_2 + \tfrac{1}{2}(\Delta\alpha_1 + \Delta\alpha_3) - \tfrac{\sqrt{3}}{2}(\Delta\beta_3 - \Delta\beta_1) \right]$$

$$m\, \Delta\ddot{\beta}_2 = 0$$

$$m\, \Delta\ddot{\alpha}_3 = -k \left[2\,\Delta\alpha_3 + \tfrac{1}{2}(\Delta\alpha_1 + \Delta\alpha_2) - \tfrac{\sqrt{3}}{2}(\Delta\beta_1 - \Delta\beta_2) \right]$$

$$m\, \Delta\ddot{\beta}_3 = 0$$

or

$$\begin{pmatrix} \Delta\ddot{\alpha}_1 \\ \Delta\alpha_2 \\ \Delta\alpha_3 \\ \Delta\beta_1 \\ \Delta\beta_2 \\ \Delta\beta_3 \end{pmatrix} = -\frac{k}{m} \begin{pmatrix} 2 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & -\tfrac{\sqrt{3}}{2} & +\tfrac{\sqrt{3}}{2} \\ \tfrac{1}{2} & 2 & \tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & -\tfrac{\sqrt{3}}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} & 2 & -\tfrac{\sqrt{3}}{2} & \tfrac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta\alpha_1 \\ \Delta\alpha_2 \\ \Delta\alpha_3 \\ \Delta\beta_1 \\ \Delta\beta_2 \\ \Delta\beta_3 \end{pmatrix}$$

In this basis, the eigenvectors found earlier are:

$$(\Delta\alpha_1, \Delta\alpha_2, \Delta\alpha_3, \Delta\beta_1, \Delta\beta_2, \Delta\beta_3)$$

$$(1,1,1,0,0,0) \qquad \lambda = 3$$

$$\left(0 \;\; -\tfrac{\sqrt{3}}{2} \;\; \tfrac{\sqrt{3}}{2} \;\; 1 \;\; -\tfrac{1}{2} \;\; -\tfrac{1}{2}\right) \qquad \lambda = 0$$

$$\left(1 \;\; -\tfrac{1}{2} \;\; -\tfrac{1}{2}, \; 0 \;\; \tfrac{\sqrt{3}}{2} \;\; -\tfrac{\sqrt{3}}{2}\right) \qquad \lambda = 0$$

$$(0 \;\; 0 \;\; 0 \;\; -1 \;\; -1 \;\; -1) \qquad \lambda = 0$$

This is a nice check that the matrix is right

It is not so hard to find 3 <u>nonzero</u> eigenvectors of the matrix

$$(a,b,c,0,0,0)$$

where (a,b,c) is an eigenvector of

$$\begin{pmatrix} 2 & \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & 2 & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} & 2 \end{pmatrix}$$

we have already find $(a,b,c) = (1,1,1) \quad \lambda = 3$

The other two are

$$(a,b,c) = (1, e^{2\pi i/3}, e^{4\pi i/3})$$

$$\lambda = 2 + \tfrac{1}{2}(e^{2\pi i/3} + e^{-2\pi i/3})$$

$$= 2 + \cos\frac{2\pi}{3} = 3/2$$

$$(a,b,c) = (1, e^{4\pi i/3}, e^{8\pi i/3})$$

$$\lambda = 2 + \tfrac{1}{2}(e^{4\pi i/3} + e^{-4\pi i/3})$$

$$= 2 + \cos\frac{4\pi}{3} = 3/2$$

check trace $= 2+2+2 = 3 + 3/2 + 3/2$ ✓

so the eigenvalues of the stability matrix are

$$\omega^2 = 3\frac{k}{m}, \ \frac{3}{2}\frac{k}{m}, \ \frac{3}{2}\frac{k}{m}$$

plus 3 required zero eigenvalues

all ω^2's are positive so the figure is stable.

3.) a.) For the sun's potential only

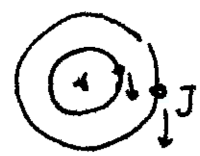
$$V = -\frac{G M_\odot m}{r}$$

$$T = \tfrac{1}{2} m v^2 = \frac{1}{2}\frac{G M_\odot m}{r}$$

$$v = \left(\frac{G M_\odot}{r}\right)^{1/2}$$

$$\text{orbital period} = t = \frac{2\pi}{(G M_\odot)^{1/2}}\, r^{3/2}$$

add the potential set up by Jupiter. I suggested that this be averaged over the orbit of Jupiter



$$V_J = \int \frac{d\phi}{2\pi} \left(- \frac{G M_J m}{\left[r^2 + r_J^2 + 2 r r_J \cos\phi\right]^{1/2}} \right)$$

Near the asteroid orbit $r = r_0$, this can be expanded

$$\cong A + B(r - r_0) + \tfrac{1}{2} C (r - r_0)^2 + \cdots$$

where

$$A \sim -\frac{G M_J m}{|r_J - r_0|} \qquad B \sim -\frac{G M_J m}{|r_J - r_0|^2} \qquad \cdots$$

write

$$B = -\frac{G M_J m}{r_0^2}\, b \qquad b \sim \frac{r_0^2}{|r_J - r_0|^2}$$

and $\quad C = + \dfrac{GM_J m}{r_0^2}\, c \qquad c \sim \dfrac{r_0^3}{(r_J - r_0)^3}$

The integral is not elementary. It has the simple representation

$$V_J = - \frac{GM_J m}{r_J} \int_0^1 \frac{dz/\pi}{[(1-z)z(1-az)]^{\frac{1}{2}}}$$

where $a = r/r_J$. As $a \to 0$ the integral $\to 1$ and $V_J \cong - \dfrac{GM_J m}{r_J}$.

To determine the shift in the orbital period, balance gravitational and centrifugal forces

$$\frac{mv^2}{r_0} = \frac{GM_\odot m}{r_0^2} + B = \frac{GM_\odot m}{r_0^2}\left(1 - \frac{M_J}{M_\odot} b\right)$$

$$v = \left(\frac{GM_\odot}{r_0}\right)^{1/2}\left(1 - \frac{1}{2}\frac{M_J}{M_\odot} b\right)$$

$$t = \frac{2\pi}{(GM_\odot)^{1/2}}\, r_0^{3/2}\left(1 + \frac{1}{2}\frac{M_J}{M_\odot} b\right) \qquad M_J/M_\odot \sim 10^{-3} \quad b \sim O(1)$$

To compute the angular velocity of the precession, follow the methods developed in class. Let $\rho = 1/r \qquad c = l/m = \text{const.}$ and

$\rho' = \dfrac{d}{d\phi}\rho \qquad$ Then

$$\rho'' + \rho = \frac{1}{\rho^2 c^2}\, f\left(\frac{1}{\rho}\right)$$

$$= \frac{GM_\odot}{c^2} - \frac{GM_J}{r_0^2 c^2}\left[b + c\frac{(r - r_0)}{r_0} + \cdots\right]\frac{1}{\rho^2}$$

Expand about $r = r_0 \qquad \rho = \rho_0 + (\rho - \rho_0)$

$$\frac{1}{\rho^2} = \frac{1}{\rho_0^2}\left(1 - 2\frac{\rho - \rho_0}{\rho_0} + \cdots\right)$$

$$(r - r_0) = -\frac{(\rho - \rho_0)}{\rho_0^2}$$

then

$$\rho'' + \rho = \frac{GM_\odot}{c^2}\left[1 - \frac{M_J}{M_\odot} b\right] + \frac{GM_\odot}{c^2}\frac{M_J}{M_\odot}(2b + c)\left(\frac{\rho - \rho_0}{\rho_0}\right) + \cdots$$

Working for small eccentricity, we can stop after this last term. It gives a small frequency shift of the oscillator

$$\rho'' + \left[1 - (2b+c)\frac{GM_\odot}{c^2}\frac{M_J}{M_\odot}\frac{1}{\rho_0}\right]\rho = (\text{constant})$$

The solution is

$$\rho = A\left(1 + e\cos\Omega(\phi - \phi_0)\right)$$

where

$$\Omega = 1 - \frac{GM_\odot}{c^2}\frac{M_J}{M_\odot}\frac{1}{\rho_0}\left(b + \frac{c}{2}\right)$$

evaluate c on the 0th order orbit

$$c^2 = (v r_0)^2 = \frac{GM_\odot}{r_0} r_0^2 = \frac{GM_\odot}{\rho_0}$$

so

$$\Omega = 1 - \frac{M_J}{M_\odot}\left(b + \frac{c}{2}\right)$$

The perihelion precesses at an angular velocity of

$$\frac{M_J}{M_\odot}\left(b + \frac{c}{2}\right) \quad \text{per radian of orbit}$$

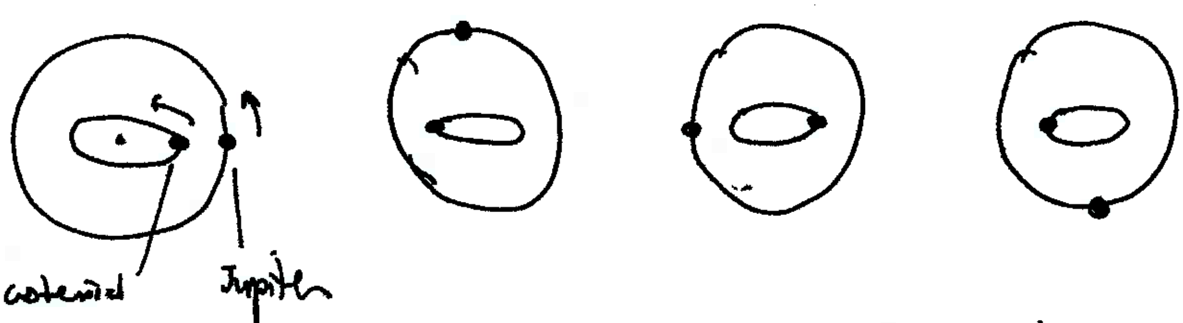
or

$$\omega = \frac{M_J}{M_\odot}\left(b + \frac{c}{2}\right)\cdot\left(\frac{GM_\odot}{r_0^3}\right)^{\frac{1}{2}}$$

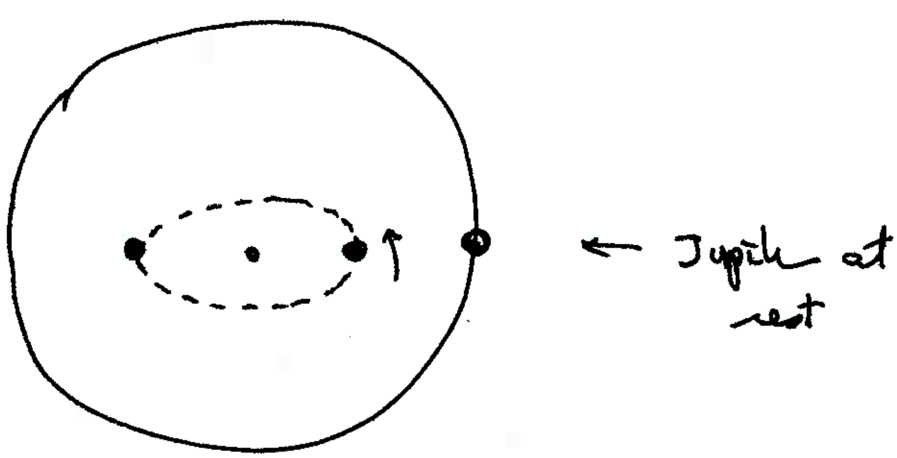
in radians per second.

b.) When the orbital periods of the asteroid and Jupiter are in an integer ratio, the gravitational perturbation due to Jupiter is in resonance with the asteroid's orbit. Then small deviations from circular motion can build up over time.

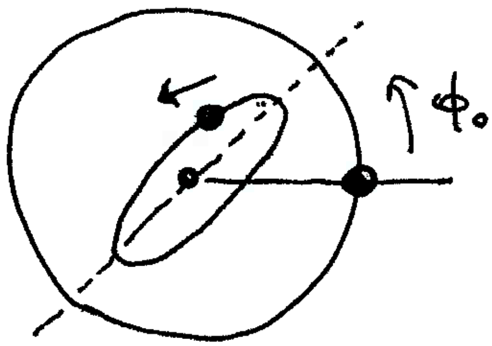
c) inertial frame:



In the frame rotating with Jupiter $\qquad \Omega_J = \left(\frac{GM_\odot}{r_J^3}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{GM_\odot}{r_0^3}\right)^{\frac{1}{2}}$



The orbit of the asteroid is a symmetrical figure with bulges in the direction of Jupiter and opposite. More generally, the asteroid's orbit might be in a general orientation:



d.) The Lagrangian for the motion of the asteroid is

$$L = \frac{m}{2}\left[(\dot{x} - \Omega_J y)^2 + (\dot{y} + \Omega_J x)^2\right] - V(r,\phi)$$

$$\dot{x} = (r\cos\phi)^{\bullet} = \dot{r}\cos\phi - \sin\phi\, r\dot{\phi}$$
$$\dot{y} = (r\sin\phi)^{\bullet} = \dot{r}\sin\phi + \cos\phi\, r\dot{\phi}$$

$$(\dot{x} - \Omega_J y)^2 + (\dot{y} + \Omega_J x)^2$$

$$= \dot{r}^2 + (r\dot{\phi})^2 - 2\Omega_J(\dot{r}\cos\phi - \sin\phi\, r\dot{\phi})\, r\sin\phi$$
$$+ 2\Omega_J(\dot{r}\sin\phi + \cos\phi\, r\dot{\phi})\, r\cos\phi + \Omega_J^2 r^2$$

$$= \dot{r}^2 + (r\dot{\phi})^2 + 2\Omega_J r^2\dot{\phi} + \Omega_J^2 r^2$$

$$= \dot{r}^2 + r^2(\dot{\phi} + \Omega_J)^2$$

so

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2(\dot{\phi}+\Omega_J)^2\right] - V(r,\phi)$$

$$\frac{V}{m} = -\frac{GM_\odot}{r} - \frac{GM_J}{[r^2+r_J^2-2rr_J\cos\phi]^{1/2}} \quad \text{(static in this frame)}$$

The conjugate momenta are $\quad p_r = m\dot{r} \qquad p_\phi = mr^2(\dot{\phi}+\Omega_J)$

so $$r^2\ddot{\phi} + 2r\dot{r}(\dot{\phi}+\Omega_J) = -\frac{1}{m}\frac{\partial V}{\partial \phi}$$

$$\ddot{r} = r(\dot{\phi}+\Omega_J)^2 - \frac{1}{m}\frac{\partial V}{\partial r}$$

First look at the case where there is no perturbation. Then

$$\frac{1}{m}V(r) = -\frac{GM_\odot}{r}$$

$$\ddot{\phi} = -2\frac{\dot{r}}{r}(\dot{\phi}+\Omega_J) \quad \text{or} \quad \frac{d}{dt}\left[r^2(\dot{\phi}+\Omega_J)\right] = 0$$

$$\ddot{r} = \frac{c^2}{r^3} - \frac{GM_\odot}{r^2}$$

where $\quad c = r^2(\dot{\phi}+\Omega_J)$

There is a circular orbit solution, as before

$$r = r_0 \qquad c^2 = GM_\odot r_0 \qquad (\dot{\phi}+\Omega_J) = \left(\frac{GM_\odot}{r_0^3}\right)^{1/2}$$

If $r = r_0 + \Delta r$

$$\Delta\ddot{r} = \left(-3\frac{c^2}{r_0^4} + 2\frac{GM_\odot}{r_0^3}\right)\Delta r = -\left(\frac{GM_\odot}{r_0^3}\right)\Delta r$$

$$\dot{\phi} = \frac{c}{r^2} - \Omega_J = \left(\frac{c}{r_0^2} - \Omega_J\right) + (\text{small})$$

so to first order, we can ignore Δr in this latter equation and write

$$\dot{\phi} = \left(\frac{GM_\odot}{r_0^3}\right)^{1/2} + O(\Delta r) - \Omega_J = \Omega - \Omega_J$$
$$= \Omega_J$$

$$\Delta r'' = \frac{d^2}{d\phi^2}\Delta r = \frac{1}{\Omega_J^2}\ddot{r}$$

so
$$\Delta r'' = -\frac{\Omega^2}{\Omega_J^2}\Delta r = -4\Delta r$$

a solution is
$$\Delta r(\phi) = a\cos(2(\phi - \phi_0))$$

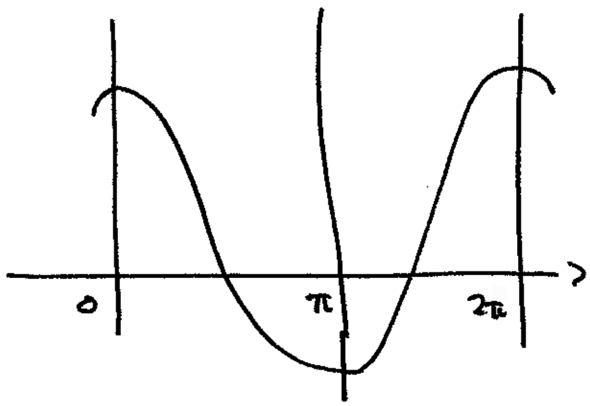
which matches the figure on p. 16.

Now add back the potential due to Jupiter

$$\Delta r'' + 4\Delta r = -\frac{1}{\Omega_J^2}\frac{GM_J}{[r^2 + r_J^2 - 2rr_J\cos\phi]^{3/2}}(r - r_J\cos\phi)$$

$$\cong + r_0 f(\phi) \quad \text{evaluate at } r = r_0$$

where $f(\phi)$ is a periodic function of the form



The equation

$$\Delta r'' + 4\Delta r = r_0 f(\phi)$$

is the equation of a harmonic oscillator subject to a periodic time-dependent force. One way to analyze its behavior is to examine the equation of energy conservation for this oscillator. Multiply the above by $\Delta r'$

$$\Delta r' (\Delta r'' + 4\Delta r) = r_0 f(\phi)$$

$$\frac{d}{d\phi}\left[\tfrac{1}{2}(\Delta r')^2 + 4(\Delta r)^2\right] = r_0 \, \Delta r'(\phi) f(\phi)$$

the chg in Energy over a cycle is

$$\oint d\phi \; r_0 \, \Delta r'(\phi) f(\phi)$$

$$= \int_0^{2\pi} d\phi \, (-2r_0 a) \sin 2(\phi - \phi_0) \, f(\phi)$$

$$= \int_0^{2\pi} d\phi \, (-2r_0 a)\left[\sin 2\phi \cos 2\phi_0 - \cos 2\phi \sin 2\phi_0\right] f(\phi)$$

$f(\phi)$ is <u>even</u> in ϕ, $\sin 2\phi$ is odd, so

$$\Delta(\text{Energy}) = \int_0^{2\pi} d\phi \; (2 r_0 a \sin 2\phi_0) \; \cos 2\phi \, f(\phi)$$

For the perturbation from Jupiter

$$\int_0^{2\pi} d\phi \; \cos 2\phi \, f(\phi) > 0$$

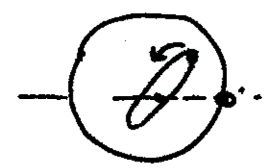
since the largest contribution to $f(\phi)$ comes near $\phi = 0$

Then

$$\Delta\text{Energy}/\text{cycle} = (\sin 2\phi_0) \cdot (\text{Positive})$$

so

for ϕ_0 between 0 and $\pi/2$



the oscillation <u>gains</u> energy; the orbit is <u>unstable</u>

for ϕ_0 between $-\pi/2$ and 0

the oscillation loses energy; the orbit reverts to a stable circular orbit

These cases are examples of the <u>stable</u> and <u>unstable</u> cycles (depend on the phase relation) that we found in our study of nonlinear Hamiltonian dynamics.