PROBLEM SET #9 SOLUTIONS - PHYSICS 210

PROBLEM 1

(a) $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are an orthonormal set.

$\hat{\beta}$ is in the osculating plane of $\vec{R}$, and $\hat{\gamma}$ is normal to it.

(b) $\frac{d\hat{\gamma}}{ds}$ is perpendicular to $\hat{\gamma}$: $\hat{\gamma}\,\frac{d\hat{\gamma}}{ds} = \frac{1}{2}\frac{d}{ds}(\hat{\gamma}\cdot\hat{\gamma}) = 0$

$\frac{d\hat{\gamma}}{ds}$ is perpendicular to $\hat{\alpha}$: $\frac{d\hat{\gamma}}{ds} = \frac{d\hat{\alpha}}{ds}\times\hat{\beta} + \hat{\alpha}\times\frac{d\hat{\beta}}{ds}$

$$= \underbrace{\cancel{-\Omega\hat{\beta}\times\hat{\beta}}}_{=0} + \hat{\alpha}\times\frac{d\hat{\beta}}{ds}$$

$$0 = \frac{d}{ds}(\hat{\beta}\cdot\hat{\gamma}) = \frac{d\hat{\beta}}{ds}\cdot\hat{\gamma} + \hat{\beta}\cdot\frac{d\hat{\gamma}}{ds} \Rightarrow \frac{d\hat{\beta}}{ds}\cdot\hat{\gamma} = \omega$$

$$0 = \frac{d}{ds}(\hat{\beta}\cdot\hat{\alpha}) = \frac{d\hat{\beta}}{ds}\cdot\hat{\alpha} + \hat{\beta}\cdot\frac{d\hat{\alpha}}{ds} \Rightarrow \frac{d\hat{\beta}}{ds}\cdot\hat{\alpha} = \Omega$$

Finally, given that $\frac{d\hat{\beta}}{ds}\cdot\hat{\beta} = 0 \Rightarrow$ $\boxed{\frac{d\hat{\beta}}{ds} = \Omega\hat{\alpha} + \omega\hat{\gamma}}$

(c) $F_3(\vec{p}, s, Y, Z) = -\vec{p}\cdot\left[\vec{R}(s) + Y\hat{\beta} + Z\hat{\gamma}\right]$

$$\Rightarrow p_s = \vec{p}\cdot\frac{d\vec{R}}{ds} = \vec{p}\cdot\hat{\alpha}\ , \quad p_Y = \vec{p}\cdot\hat{\beta}\ , \quad p_Z = \vec{p}\cdot\hat{\gamma}$$

So H looks the same in the new system, with $\vec{p}$ and $\vec{A}$ given in the $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ basis.

(d) Use $\vec{A} = \left(\frac{B_0 r}{2}\right)\hat{\phi} = \frac{B_0}{2}(R_0 - Y)\hat{\alpha}$

$$\Rightarrow \quad H = \frac{1}{2m}\left(\vec{p} - \frac{qB_0}{2c}(R_0 - Y)\hat{\alpha}\right)^2 , \quad \begin{cases} \hat{\gamma} = \hat{z} \\ \hat{\beta} = -\hat{r} \\ \hat{\alpha} = \hat{\phi} \end{cases}$$

For the circular orbit at radius R_0,

$$p_s = m\omega_c R_0 = \frac{qB_0R_0}{c} , \quad s = \omega_c R_0 t , \quad Y = Z = p_Y = p_Z = 0$$

(e) $\vec{p} = \vec{\mathbb{P}} + \frac{qB_0R_0}{c}\hat{\alpha} \Rightarrow H = \frac{1}{2m}\left(\vec{\mathbb{P}} + \frac{qB_0}{2c}(R_0 + Y)\hat{\alpha}\right)^2$

$$= \frac{1}{2m}\left(\vec{\mathbb{P}}^2 + \frac{qB_0}{c}(R_0 + Y)\mathbb{P}_s + \left(\frac{qB_0}{2c}\right)^2 (R_0 + Y)^2\right)$$

So the eq. of motion are:

$$\frac{d}{dt}\begin{pmatrix} s \\ Y \\ Z \\ \mathbb{P}_s \\ \mathbb{P}_Y \\ \mathbb{P}_Z \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 0 & qB_0/2c & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(qB_0/2c)^2 & 0 & -qB_0/2c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s \\ Y \\ Z \\ \mathbb{P}_s \\ \mathbb{P}_Y \\ \mathbb{P}_Z \end{pmatrix}$$

The eigenvectors with $\lambda = 0$ are

$(1,0,0,0,0,0)$: free displacement along the orbit

$(0,0,1,0,0,0)$: " " in $\hat{z}$

$(0,-1,0,\frac{qB_0}{2c},0,0)$: change of radius

$(0,0,0,0,0,1)$: Constant motion in the $\hat{z}$ direction

The other eigenvalues / eigenvectors are

$\lambda = \pm \frac{i\, qB_0}{2mc}$ for $\left(-1, \pm i, 0, 0, \frac{qB_0}{2c}, 0\right)$,

corresponding to change in the center of the orbit.

(f) To 1st order : $\frac{\partial \vec{B}}{\partial z} = \frac{-n\hat{r}}{R_0}$, $\frac{\partial \vec{B}}{\partial r} = -\frac{n}{R_0}\hat{z}$ $\Rightarrow$ $\vec{\nabla}\times\vec{B} = 0$

(g) A choice of vector potential that satisfies $\vec{\nabla}\times\vec{A} = \vec{B}$ is

$$A_\varphi = B_0\left(\frac{n}{R_0}\left(\frac{z^2}{2} - \frac{r^2}{3}\right) + \left(\frac{1+n}{2}\right)r\right)$$

$$\Rightarrow H = \frac{1}{2m}\left(\vec{p} + \frac{qB_0R_0}{c}\hat{\alpha} - \frac{qB_0}{c}\left(\frac{n}{R_0}\left(\frac{z^2}{2} - \frac{(R_0-Y)^2}{3}\right) + (1-n)\frac{(R_0-Y)}{2}\right)\hat{\alpha}\right)^2$$

$$\Rightarrow \left.\frac{\partial^2 H}{\partial p_3\, \partial Y}\right|_0 = \frac{1}{m}\frac{qB}{2c}\left(1-\frac{n}{3}\right) , \quad \left.\frac{\partial^2 H}{\partial Y^2}\right|_0 = \frac{1}{m}\left(\frac{qB}{2c}\right)^2\left(1+\frac{2n}{3}-\frac{n^2}{3}\right)$$

$$\left.\frac{\partial^2 H}{\partial z^2}\right|_0 = \frac{1}{m}\left(\frac{qB}{2c}\right)^2 2n\left(1-\frac{n}{3}\right)$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} s \\ y \\ z \\ p_s \\ p_y \\ p_z \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 0 & \frac{qB}{2c}(1-n/3) & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\left(\frac{qB}{2c}\right)^2\left(1+\frac{2n}{3}-\frac{n^2}{3}\right) & 0 & -\frac{qB}{2c}\left(1-\frac{n}{3}\right) & 0 & 0 \\ 0 & 0 & \left(\frac{qB}{2c}\right)^2 2n\left(1-\frac{n}{3}\right) & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s \\ y \\ z \\ p_s \\ p_y \\ p_z \end{pmatrix}$$

The eigenvalues are $0, \ \pm\frac{qB}{2mc}\sqrt{-\frac{2}{3}n(n-3)}, \ \pm\frac{qB}{2mc}\sqrt{\frac{1}{3}(n+1)(n-3)}$

For $-1 < n < 0$, there are no unstable perturbations.

PROBLEM 2

(a) $f^2 = (4\lambda)^2 x(1-x)\left(1-4\lambda x(1-x)\right)$

So $f^2(x) = x \Rightarrow x = \frac{1}{8\lambda}\left(4\lambda+1 \pm \sqrt{(4\lambda-3)(4\lambda+1)}\right)$

$$f^{2'}(x) = \left.\frac{d}{dx} f(f(x))\right|_x = \left. f'(x)\right|_{f(x)} \left. f'(x)\right|_x$$

Given that

$f'(x) = 4\lambda(1-2x) = -1 \pm \sqrt{(4\lambda-3)(4\lambda+1)}$, we have:

$f^{(2)'}(x) = 1 - (4\lambda-3)(4\lambda+1) = -16\lambda^2 + 8\lambda + 4$

So $f^{(2)'}(x) = -1 \Rightarrow 0 = 16\lambda^2 - 8\lambda - 5$

$$\Rightarrow \boxed{\lambda_3 = \frac{1}{4}(1+\sqrt{6})}$$

(b)

0.5	stable fixed point $x = 0.5$
0.73	" " " $x = 0.6575$
0.86	" 2-cycle $x = 0.4422, 0.9447$
0.89	" 8-cycle
0.90	chaos
0.93	chaos
0.958	stable 3-cycle $x = 0.1542, 0.4995, 0.9575$
0.97	chaos